# Various Variants of Wythoff Nim

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**Abstract**

We describe a unified view of a broad class of generalizations of Wythoff Nim. If the $P$-positions lie at $o(n)$ distance from a set of lines, then the slopes and relative densities of $P$-positions on those lines are a solution to a specific set of rational equations; in particular, the lines have algebraic slope. We compute the minimal polynomials for some cases of interest, e.g. $(1,2)$GDWN. Using this idea, we prove one direction of a conjecture of Larsson about "splitting pairs." In the case that the $P$-positions are at $\Omega(n)$ distance from lines, such as what appears to happen for $(3,5)$GDWN, we give intuition for the apparent quasiperiodicity by a connection to intransitive dice. Finally, we outline the first half of a proof of a conjecture of Dekking, et al. about the $P$-positions of the greedy placement of queens in the first quadrant of the plane.

## 1 Introduction

Nim is perhaps the most fundamental and well-studied combinatorial game. The rules are simple: at any stage of the game there are some number of stones arranged in piles, a move is to remove any number of stones from a single pile, and the winner is the player that removes the last stone. The game where all the piles have no stones is understood to be a second-player win. It is well-known that the game is a second-player win if and only if the so-called *nim-sum* of the number of stones in the piles is 0. This nim-sum is the bitwise xor operation. In the case of 2-pile Nim, this means the game is a second-player win if and only if the piles start with an equal number of stones; in that case, the optimal play for the second player is to equalize the number of stones in the two piles (i.e. copy the first player's move on the opposite pile). In general, a position in an impartial game such as Nim is called a $P$-position if it is a win for the second player and an $N$-position if it is a win for the first player ("P" stands for "previous player" and "N" stands for "next player"). The set of $P$-positions for a game is denoted $\mathcal{P}$ and the set of N-positions is denoted $\mathcal{N}$.

*Wythoff Nim*, first defined in [Wyt07], is a classic variant of 2-pile Nim where, in addition to the ability to take any number of stones from a single pile, one may also take an equal number of stones from both piles. This drastically changes the set of $P$-positions. This is intuitive: all of the old $P$-positions are now instantaneous wins for the first player. Wythoff proved that the $P$-positions in fact lie nearly exactly on two lines; namely the $P$-positions are

$$(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) \qquad \text{and} \qquad (\lfloor \phi^2 n \rfloor, \lfloor \phi n \rfloor)$$

for each $n \geq 0$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and $(x, y)$ means the first pile has $x$ stones and the second has $y$ stones. Wythoff's proof of this essentially relies on the famous property of *complementary Beatty sequences*, which is that if $1/a + 1/b = 1$ with $a, b$ positive irrational numbers, then the sequences

$$(\lfloor an \rfloor)_{n=1}^{\infty} \qquad \text{and} \qquad (\lfloor bn \rfloor)_{n=1}^{\infty}$$

form a partition of the positive integers. Alternative proofs may be obtained by properties of the *Fibonacci word*, which is defined by repeatedly applying the string substitution $0 \to 1$ and $1 \to 10$ to the word. The interpretation of this word is that the 0's correspond to a $P$-position on the lower line and the 1's correspond to a $P$-position on the upper line, where the $P$-positions are read from left to right.

Many variants of Wythoff Nim have been studied; see [Duc+17]. In some cases, Beatty sequences or more complicated string substitutions can be used to analyze the asymptotic $P$-position placement in variants, but general analysis is difficult. In this paper we concern ourselves foremost with *Generalized Diagonal Wythoff Nims* (GDWNs), introduced by Larsson in [Lar12]. Larsson defines

$$\{(p_1, q_1), (p_2, q_2), \dots\}\text{GDWN}$$

to be the 2-pile Nim variant where the allowed moves are to remove any number of stones from a single pile, an equal number of stones from both piles, or $p_i n$ stones from one pile and $q_i n$ notes from the other pile, for some $i, n \geq 1$. Often braces and commas are omitted, e.g. $(1,2)(2,3)$GDWN is $\{(1,2),(2,3)\}$GDWN. In fact we consider a more general form. We define

$$\{(p_1, q_1), (p_2, q_2), \dots\}\text{GDN}_k$$

to be the 2-pile Nim variant where the allowed moves are to remove any number of stones from a single pile, $kn$ stones from both piles where $n \geq 1$, or $p_i n$ stones from the *first* pile and $q_i n$ stones from the *second* pile, for some $i, n \geq 1$. If $k$ is omitted it is understood to equal 1, and as before one may omit the braces and commas. Note that in these variants the moves are no longer symmetrical between the piles in general, e.g. $(1,2)$GDN is different than $(1,2)(2,1)$GDN. The reason to require that some move of the form $(k,k)$ is allowed is because otherwise it is easily shown that the $P$-positions are the same as in Nim, namely $\{(n,n)\}_{n=0}^{\infty}$. We describe this argument in Section 2. Also, we can allow the $q$'s to be negative, with the interpretation being that a move of that form adds stones to the second pile. Since the number of stones in the first pile either decreases or stays the same, and in the event it stays the same the number of stones in the second pile decreases, the game will still always end in finitely many moves.

Intuitively, if $b/a$ is approximately $\phi$, then $(a,b)$GDN (or indeed $(a,b)$GDWN) should have substantially different $P$-positions than Wythoff Nim. More precisely, Larsson defines $(p,q)$ to be a *splitting pair* if either $(p,q)$ or $(p-1,q-1)$ is a $P$-position of Wythoff Nim other than $(0,0)$. He makes the following conjecture:

**Conjecture 1.** *Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ be the subset of $P$-positions of $(p,q)$GDWN such that $a_1 \leq a_2 \leq \dots$ and $b_n < a_n$ for all $n$. Then $\lim_{n\to\infty} \frac{b_n}{a_n} = \phi$ if and only if $(p,q)$ is not a splitting pair.*

Larsson proved the "if" direction when $p < q < \phi p$ and the "only if" direction for the specific cases $(p,q) = (1,2)$ and $(p,q) = (2,3)$ in [Lar12]. We prove this conjecture for all splitting pairs (i.e. the entirety of the "only if" direction), leaving open only the case when $q > \phi p$ and $(p,q)$ is not a splitting pair. In fact we prove something stronger in Section 3, namely:

**Theorem 1.** *Under a suitable assumption (that is known to be satisfied for Wythoff Nim),*

*(1) There is a finite set $S$ of real numbers such that if $\mathcal{P}$ are the $P$-positions of $\mathcal{D}$GDN$_k$ then the function $E$ given by $E(a) = \min_{s\in S}\{|b - sa|\}$ is $o(a)$ for almost all $(a,b) \in \mathcal{P}$.*

*(2) There is a set of real numbers $\Delta = \{\delta_s\}_{s\in S}$ such that the proportion of $P$-positions satisfying $|b - sa| \leq E(a)$ for a particular $s$ approaches $\delta_s$ as $a \to \infty$.*

*(3) There is a set $D \subseteq \mathcal{D} \cup \{(0,1),(1,0),(k,k)\}$ with $|D| = |S| + 1$ such that for all $(c,d) \in D$,*

$$\sum_{\substack{s\in S\\ s>d/c}} \frac{\delta_s}{cs - d} = 1 \qquad \textit{and} \qquad \sum_{\substack{s\in S\\ s<d/c}} \frac{\delta_s}{d - cs} = 1$$

*with $1/0 = \infty$, except for $(c,d) = (0,1)$ or $(1,0)$ where one of the above equalities holds and the other sum is empty. Furthermore one may order the $S$ as $s_1, s_2, \dots$ so that if $D$ is ordered $(c_1,d_1),(c_2,d_2),\dots$ so that $(c,d)$ comes before $(c',d')$ whenever $d/c < d'/c'$ (with $1/0 = \infty$) then*

$$0 = \frac{d_1}{c_1} < s_1 < \frac{d_2}{c_2} < s_2 < \dots < s_{|S|} < \frac{d_{|D|}}{c_{|D|}} = \infty.$$

*(4) If either*

$$\sum_{\substack{s\in S\\ s<q/p}} \frac{\delta_s}{q-ps} > 1 \qquad \text{or} \qquad \sum_{\substack{s\in S\\ s>q/p}} \frac{\delta_s}{ps-q} > 1$$

*then whenever $(p,q) \in \mathcal{E}$, the function given by $\min_{s\in S}\{|b-sa|\}$ is not $o(n)$, where $\{(a,b)\}$ are the $P$-positions of $(\mathcal{D}\cup\mathcal{E})\,GDN_k$.*

See Section 3 for the details of the assumption. The set $S$ in the theorem above is called the set of *slopes*, $\Delta$ the set of *densities*, and $D$ the set of *dominant vectors*. A $(p,q)$ satisfying either inequality in part (4) is called a *splitting vector*.

We further conjecture:

**Conjecture 2.** *Under the assumptions of Theorem 1, if*

$$\sum_{\substack{s\in S\\ s<q/p}} \frac{\delta_s}{q-ps} \le 1 \qquad \text{and} \qquad \sum_{\substack{s\in S\\ s>q/p}} \frac{\delta_s}{ps-q} \le 1,$$

*then $(\mathcal{D}\cup\{(p,q)\})\,GDN_k$ also satisfies the assumptions of Theorem 1 and the set of slopes, densities, and dominant vectors is the same as in $\mathcal{D}\,GDN_k$.*

This would imply the "if" direction of Conjecture 1.

This paper is organized as follows. In Section 2 we describe a simple (though slow) algorithm to compute the $P$-positions in variants of Nim. This shows some basic properties of the $P$-position placement. In Section 3 we prove Theorem 1 then perform many computations using parts (3) and (4) and some generalizations of them. In Section 4 we provide some intuition for the behavior of GDNs that do not satisfy the assumptions of Theorem 1, though there are no proofs. There appears to be a connection to a suitably generalized version of intransitive dice. Finally, in Section 5, we outline a proof of a conjecture in [DSS19] that the $P$-positions of $(1,-1)$GDN lie at bounded distance from the lines of slopes $\phi$ and $1/\phi$ through the origin.

# 2 Computing P-Positions with Fairy Chess Pieces

How do you determine the $P$-positions of a given GDN? Since the position always decreases lexicographically over the course of a game, we will consider the possible positions in lexicographically-increasing order, starting from $(0,0)$ (which is a $P$-position). Note that a position is an $N$-position if and only if there is a legal move to a $P$-position, and it is a $P$-position if and only if all legal moves go to $N$-positions (including vacuously the case where there are no legal moves). This remark gives an algorithm for determining the $P$-positions:

(1) Place a *fairy* at $(0,0)$.

(2) In lexicographically-increasing order (i.e. upwards along columns), if a position is not "seen" by any previously placed fairy, place a fairy at that position.

(3) The $P$-positions are precisely the positions of the fairies.

Here, "fairy" refers to a generalized chess piece depending on the variant. For regular Nim, the fairies are rooks. For Wythoff Nim, they are rooks that can also move along diagonals (but not antidiagonals), a sort of "one-eyed queen." For $(1,-1)$GDN, they are queens. This variant is of interest for precisely this reason; the $P$-positions of $(1,-1)$GDN are known as the "greedy queens" sequence [DSS19]. For $(1,2)$GDWN, they are (one-eyed) "amazonriders." And so on. The name "fairy" is coming from "fairy chess," which are variants of chess with different rules, particularly the introduction of "fairy chess pieces," pieces with different movement from the pieces in standard chess.

From this algorithm, it is clear that the $P$-positions of a variant of Nim are $(n,n)$ for all $n\ge 0$ if and only if no move of the form $(a,a)$ is allowed. This is the reason for appending the move $(1,1)$ in the definition of GDWN and $(k,k)$ in $\mathrm{GDN}_k$. More generally, if one adds a move $(a,b)$ to a variant of Nim, the $P$-positions do not change if and only if no two $P$-positions differed by $(a, b)$. This was the key behind the proof in [Lar12] of the "if" direction of Conjecture 1 in the case $p < q < \phi p$.

It is also the case that every row and column will contain a $P$-position in any GDN. For columns this is immediate. For rows, notice that for large enough $x$, all points $(x, n)$ for fixed $n$ will see no fairies, at least in the case that no legal move increases the $y$-coordinate, i.e. none of the $q$'s are negative. This is still true in the case when there are moves that increase the $y$-coordinate, and we invite the reader to ponder.

Also, if no legal move increases the $y$-coordinate then not only is the position decreasing lexicographically, but it is in fact decreasing under the $L^1$ norm. Hence in that case you can modify the algorithm by considering the squares going down along antidiagonals rather than up along columns and you will get the same result. In fact, as long as you consider the squares along sufficiently steep antidiagonals, you will get the same result as considering them going up columns; this corresponds to considering the states in increasing order according to the norm $\|(x, y)\|_c = c|x| + |y|$ for some $c$. In a way, going "up columns" is the same as "down infinitely steep diagonals," as the lexicographical ordering is obtained by taking $c \to \infty$.

Finally, if you forget about Nim and just deal with fairies to begin with, you can define further variants on other spaces or orderings. For instance, in $\mathbb{Z}^2$, place pieces starting at the origin and moving outwards in a square spiral; in $\{(x, y) \in \mathbb{N}^2 \mid x \geq y\}$, place pieces upwards on columns; in $\mathbb{N}^2$, place pieces alternatively going up and down antidiagonals (boustrophedon); and so on. Many of the results and conjectures have obvious extensions for these other ordered spaces. In Section 3.2.4 we consider the queens on a square spiral in $\mathbb{Z}^2$, and we invite the reader to further generalize. There are ways to obtain Nim variants for these more exotic variants by just adding the rule that the state must decrease under the chosen ordering so that the game always ends, but this does not seem to be a particularly useful way to view these variants.

# 3 A Heuristic for Slopes of Lines

## 3.1 Proof of Theorem 1

We must introduce some notation. Write $\tilde{\mathcal{D}} = \mathcal{D} \cup \{(0, 1), (1, 0), (k, k)\}$. For $(c, d) \in \tilde{\mathcal{D}}$, $j \in \mathbb{Z}$, and $p \in \{0, \ldots, c-1\}$, define $\ell(c, d, j, p)$ as

$$\ell(c, d, j, p) = \begin{cases} \{(nc + p, nd + j) \mid n \in \mathbb{Z}\} & (c, d) \neq (0, 1) \\ \{(j, n) \mid n \in \mathbb{Z}\} & (c, d) = (0, 1) \end{cases}.$$

In the case $(c, d) = (0, 1)$ we suppress $p$ and just write $\ell(0, 1, j)$. Let $H(c, d, j, p)$ be the column that contains the $P$-position in $\ell(c, d, j, p)$ (by assumption there is at most one $P$-position in each $\ell(c, d, j, p)$) or $\infty$ if there is no $P$-position in $\ell(c, d, j, p)$.

Call $(c, d) \in \tilde{\mathcal{D}}$ *nice* if either:

- There are constants $\alpha$ and $\beta$ such that

  $$H(c, d, j, p) = \alpha j + o(j)$$

  for negative $j$ and

  $$H(c, d, j, p) = \beta j + o(j)$$

  for positive $j$. In the case $(c, d) = (0, 1)$ or $(1, 0)$ we only require $\beta$ to exist since all $P$-positions are in the first quadrant.

- Otherwise, a positive fraction of $\ell(c, d, j, p)$ do not contain a $P$-position, taking this fraction as $|j| \to \infty$.

The $(c, d)$ of the first type will end up constituting the $D$ claimed in part (3). Call $\mathcal{D}\mathrm{GDN}_k$ *nice* if all $(c, d) \in \tilde{\mathcal{D}}$ are nice.

The main intuition behind the first half of this theorem is that, if $\mathcal{D}\mathrm{GDN}_k$ is nice, then after placing the first $n$ $P$-positions there is a "block" of lines of slope $d/c$ such that all lines in this block have one $P$-position, and this block grows linearly in $n$. Then the next few $P$-positions will occur both outside of these blocks and as far down as possible, which is near the vertex of a wedge in the plane formed by the space in-between the blocks. This vertex moves along a line as the blocks grow. The block growth rate, and therefore the slope of

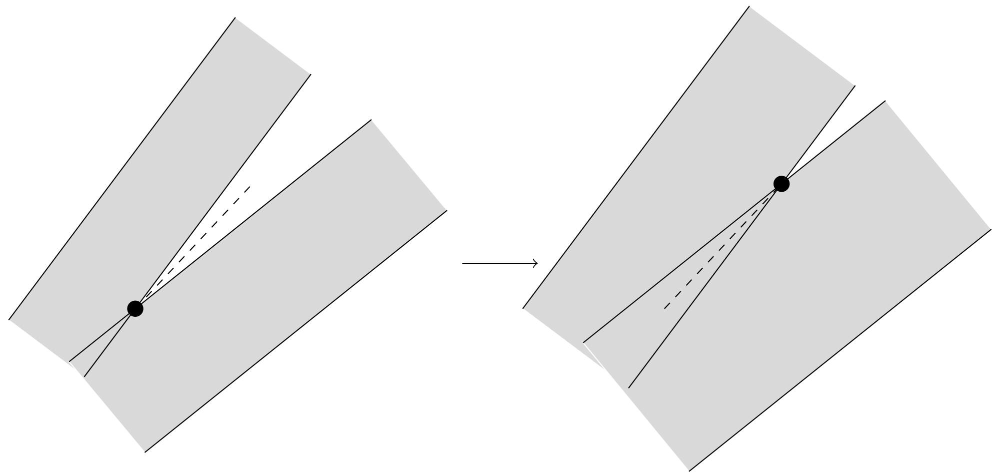

Figure 1: Intuition for Theorem 1. The blocks are regions where there cannot be any $P$-positions due to a single $(c,d) \in \tilde{\mathcal{D}}$. They grow linearly, pushing the vertex along a line. In general there will be more blocks. The $P$-positions are always placed near a vertex between two blocks, so they should lie near some set of lines. The angle of the blocks relative to the horizontal is determined by $d/c$, so combined with the growth rates of the blocks, this determines the density and slope of each line.

the line this vertex moves along, is determined by the $\alpha$ and $\beta$ values, and this is quantified in the system of equations in part (3). See Figure 1.

We note the following fact about nice games. Define $L(\alpha, \beta)$ as

$$L(\alpha,\beta) = \begin{cases} \{\{(x+nc, y+nd) \mid n \in \mathbb{Z}\} \mid 0 \le x < nc, \alpha \le y \le \beta\} & (c,d) \notin \{(0,1),(1,0)\} \\ \{\{(n,y) \mid n \in \mathbb{Z}\} \mid \alpha \le y \le \beta\} & (c,d) = (1,0) \\ \{\{(-x,n) \mid n \in \mathbb{Z}\} \mid \alpha \le x \le \beta\} & (c,d) = (0,1). \end{cases}$$

Let $\mathcal{P}(n) = \mathcal{P} \cap ([0,n] \times [0,\infty))$. Then the condition that $(c,d) \in \tilde{\mathcal{D}}$ is nice implies the existence of $d_\alpha(c,d)$ and $d_\beta(c,d)$ such that $d_\alpha(c,d) \le 0 \le d_\beta(c,d)$ and

$$|L(d_\alpha(c,d)n, d_\beta(c,d)n) \cap \mathcal{P}(n)| = c(d_\beta(c,d) - d_\alpha(c,d))n + o(n)$$

but for all $d'_\alpha, d'_\beta$ such that either $d'_\alpha = d_\alpha(c,d)$ and $d'_\beta > d_\beta(c,d)$ or $d'_\beta = d_\beta(c,d)$ and $d'_\alpha < d_\alpha(c,d)$, there is an $\varepsilon > 0$ such that

$$|L(d'_\alpha n, d'_\beta n) \cap \mathcal{P}(n)| \le (1-\varepsilon)c(d'_\beta - d'_\alpha)n + o(n)$$

for all $n$. This is what we will use to prove Theorem 1. In the case of nice $(c,d)$ of the second type, $d_\alpha = d_\beta = 0$.

Note that Wythoff Nim is nice, which follows quickly from the closed form expression for the $P$-positions. In fact in basically every known variant that has a proof of some asymptotic behavior of $P$-positions is nice by virtue of that proof itself.

We now present the proof of Theorem 1, though we are slightly informal in places.

*Proof of Theorem 1.* Consider first the niceness of $(1,0)$. Obviously $d_\alpha(1,0) = 0$. For notational simplicity set $d_\beta = d_\beta(1,0)$. Niceness tells us that the $(d_\beta n)$th row (rounded down, say) above the $x$-axis must contain a $P$-position no later than column $n + o(n)$, so the $P$-positions are almost all bounded below by the line of slope $d_\beta$ through the origin. At the same time, niceness tells us that for any $d'_\beta > d_\beta$, a positive fraction of rows between $d_\beta n$ and $d'_\beta n$ above the $x$-axis do not have $P$-position earlier than column $n + o(n)$, so there must be a $P$-position with $y$-coordinate between $d_\beta n$ and $d'_\beta n$ and $x$-coordinate at least $n + o(n)$, for any $d'_\beta > d_\beta$ and $n \ge 0$. Combined with the lower bound, this implies that a positive fraction $r_{1,0}$ of $P$-positions must lie at $o(n)$ distance from the line of slope $d_\beta$ through the origin, and for any $d'_\beta > d_\beta$ but sufficiently close to $d_\beta$, there are almost no $P$-positions at $o(n)$ distance from the line of slope $d'_\beta$ through the origin.

Consider the next smallest $(c, d) \in \tilde{\mathcal{D}}$, ordered by the value of $d/c$, such that the associated $d_\beta(c, d)$ is nonzero. First witness that $d_\alpha(c, d) = d_\beta(1, 0) - d/c$. Indeed, almost no $P$-positions lie below the line of slope $d_\beta(1, 0)$, so for $d'_\alpha < d_\beta(1, 0) - d/c$ and $d'_\beta = d_\beta(c, d)$ the second condition for niceness will be satisfied. This implies $d_\alpha(c, d) \geq d_\beta(1, 0) - d/c$. The fact that there are almost no $P$-positions at $o(n)$ distance from the line of slope $d_\beta(1, 0) + \varepsilon$ for sufficiently small $\varepsilon$ implies that $d_\alpha(c, d) \leq d_\beta(1, 0) - d/c$.

Now niceness tells us that the lines $\{(x + nc, \lfloor d_\beta(c, d)n \rfloor + nd) \mid n \in \mathbb{Z}\}$ for $0 \leq x < c$ each have a $P$-position no later than column $n + o(n)$. This means other than the fraction $r$ of $P$-positions at $o(n)$ distance from the line of slope $d_\beta(1, 0)$, almost all $P$-positions lie above the line of the slope $d_\beta(c, d) + d/c$. Similarly to the case of rows, niceness also tells us that for any $d'_\beta > d_\beta(c, d)$ there must be a $P$-position on one of the lines $\{(x + nc, y + nd) \mid n \in \mathbb{Z}\}$ with $d_\beta(c, d)n \leq y \leq d'_\beta n$, such that the $P$-position has $x$-coordinate $n + o(n)$. Thus a positive fraction $r_{c,d}$ of $P$-positions must lie at $o(n)$ distance from the line of slope $d_\beta(c, d) + d/c$.

This argument continues for all such $(c, d) \in \tilde{\mathcal{D}}$ with $d_\alpha(c, d)$ and $d_\beta(c, d)$ not both zero, until $(c, d) = (0, 1)$. We omit the details. Let $(c^*, d^*)$ be the last such $(c, d)$ before $(0, 1)$. Then we find by the fact that fairy pieces are placed as low as possible that almost no $P$-positions lie $\Omega(n)$ distance *above* the line of slope $d_\beta(c^*, d^*)$. Also note $d_\alpha(0, 1) = -1$ and $d_\beta(0, 1) = 0$.

Now we prove the specific claims of the theorem:

(1) The set $S$ is $\{d_\beta(c, d) + d/c \mid (c, d) \in \tilde{\mathcal{D}}$ such that $d_\beta(c, d) \neq 0\}$, as we have seen.

(2) Define $P_s(N)$ to be the number of $P$-positions distance $o(n)$ from the line of slope $s$. We have seen $\lim_{N\to\infty} P_s(N)/N = r_{c,d}$ where $s = d_\beta(c, d) + d/c$, and $\delta_s$ is also the value of this limit.

(3) The set is $D = \{(c, d) \in \tilde{\mathcal{D}} \mid d_\beta(c, d) \neq 0\} \cup \{(0, 1)\}$. The inequality chain

$$0 = \frac{d_1}{c_1} < s_1 < \frac{d_2}{c_2} < s_2 < \cdots < s_{|S|} < \frac{d_{|D|}}{c_{|D|}} = \infty.$$

is clear.

Consider $(c, d) \in D$ other than $(0, 1)$. Almost all lines of the form $\{(x + nc, y + nd) \mid n \in \mathbb{Z}\}$ have a $P$-position (except in the case $(c, d) = (1, 0)$ with $y < 0$, where none do). Let $\mathcal{P}' = \{(x, y - \lfloor xd/c \rfloor) \mid (x, y) \in \mathcal{P}\}$. Let $s^*$ be the smallest element of $S$ that is greater than $d/c$. We will compute the size of $X(N) = \mathcal{P}' \cap [0, N] \times [0, (s^* - d/c)N]$ in two ways.

First, there one $P$-position on almost all sets of the form $\{(x + nc, y + nd) \mid n \in \mathbb{Z}\}$, as we mentioned. Thus there are $\lfloor c(s^* - d/c)N \rfloor$ elements of $X(N)$ up to an $o(N)$ term, since those sets are sheared to now be the sets $\{(x + nc, y - \lfloor xd/c \rfloor) \mid n \in \mathbb{Z}\}$ and there are $\lfloor c(s^* - d/c)N \rfloor$ such sets covering $([0, N] \times [0, (s^* - d/c)N]) \cap \mathbb{Z}^2$. Second, $X(N)$ contains almost all of the $P$-positions $(x, y)$ with $dy > cx$ and $x \leq N$, so since $P_s(N)/N \to \delta_s$ for all $s$, we have

$$|X(N)| = \sum_{\substack{s \in S \\ s > d/c}} \frac{s^* - d/c}{s - d/c} \delta_s N + o(N).$$

This is just geometry. Consider Figure 2 depicting $X(N)$, where $S \cap (d/c, \infty) = \{s^*, s_1, s_2, s_3\}$ with $s^* < s_1 < s_2 < s_3$ for example.

Note that after shearing the $P$-positions to obtain $X(N)$, the slopes of the diagonal lines are not $s^*, s_1, s_2, s_3$ but $s^* - d/c$, etc. Up to $o(N)$, the number of $P$-positions inside the rectangle is $\delta_{s^*} N + \frac{s^* - d/c}{s_1 - d/c} N \ldots$ as desired.

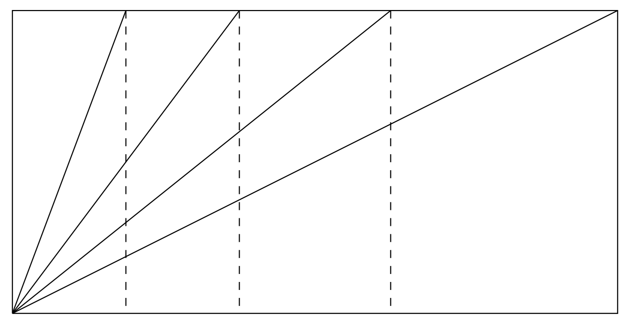


Figure 2: Geometric argument for parts (3) and (4) of Theorem 1. This depicts the rectangular region containing $X(N)$. The $P$-positions essentially lie on the lines of slopes $s^*, s_1, s_2, s_3$ with constant density, so we can compute the number of $P$-positions in $X(N)$ by finding the size of the line segments in this rectangle.

Therefore

$$\sum_{\substack{s\in S\\ s>d/c}} \frac{s^*-d/c}{s-d/c}\delta_s N + o(N) = \lfloor c(s^*-d/c)N\rfloor + o(N)$$

$$= c(s^*-d/c)N + o(N)$$

$$\sum_{\substack{s\in S\\ s>d/c}} \frac{\delta_s}{cs-d} = 1 \qquad \text{(divide by } c(s^*-d/c)N \text{ and take } N\to\infty).$$

A similar calculation for the size of $\mathcal{P}'\cap[0,N]\times[0,(-s_*+d/c)N]$ where $s_*$ is the largest element of $S$ less than $d/c$ gives

$$\sum_{\substack{s\in S\\ s<d/c}} \frac{\delta_s}{d-cs} = 1$$

except for $(c,d)=(1,0)$ where this sum is empty.

For $(c,d)=(1,0)$, $\sum_{s\in S}\delta_s = 1$ since almost all $P$-positions lie $o(N)$ distance from the lines with slopes in $S$. The other sum is $\sum_{\substack{s\in S\\ s>1/0}}\delta_s$, which is empty.

(4) We use a similar geometric argument to the previous part. Let $\mathcal{P}' = \{(x, y-\lfloor xq/p\rfloor) \mid (x,y)\in\mathcal{P}\}$. We just prove the

$$\sum_{\substack{s\in S\\ s>q/p}} \frac{\delta_s}{ps-q} > 1$$

case by considering $X(N) = \mathcal{P}'\cap[0,N]\times[0,(s^*-q/p)N]$ with $s^*$ the smallest element of $S$ greater than $q/p$; the other is essentially the same by looking at $\mathcal{P}'\cap([0,N]\times[0,(-s_*+q/p)N])$ with $s_*$ the largest element of $S$ less than $q/p$. Note that in $(\mathcal{D}\cup\mathcal{E})\mathrm{GDN}_k$ where $(p,q)\in\mathcal{E}$ that each line of the form $\{(x+np, y+nq)\mid n\in\mathbb{Z}\}$ has at most one $P$-position. After shearing, this means there are at most $\lfloor p(s^*-q/p)N\rfloor$ elements of $X(N)$. Thus after considering Figure 2 once more, if the $P$-positions of $(\mathcal{D}\cup\mathcal{E})\mathrm{GDN}_k$ were $o(n)$ from lines with slopes in $S$, we would have

$$\sum_{\substack{s\in S\\ s>q/p}} \frac{s^*-q/p}{s-q/p}\delta_s N + o(N) \le \lfloor p(s^*-q/p)N\rfloor$$

$$\le p(s^*-q/p)N$$

$$\sum_{\substack{s\in S\\ s>q/p}} \frac{\delta_s}{ps-q} \le 1,$$

contradicting that $\sum_{\substack{s\in S\\ s>q/p}} \frac{\delta_s}{ps-q} > 1$. There is a small wrinkle since it could be the case that $(\mathcal{D}\cup\mathcal{E})\mathrm{GDN}_k$ is not nice but still has all $P$-positions $o(n)$ from lines with slopes in $S$, so long as there are exponentially-growing regions of high and low density along these lines (see Figure 5 for something similar to this phenomenon). But in that case one simply needs to consider the lim sup of $|X(N)|$, which must still be at least $\sum_{\substack{s\in S\\ s>q/p}} \frac{s^*-q/p}{s-q/p}\delta_s N + o(N)$.

□

We remark that if niceness is replaced by the same condition but with $O(1)$ instead of $o(n)$, an identical proof replacing $o(n)$ (or $o(N)$) with $O(1)$ everywhere shows that the $P$-positions are at a bounded distance from a set of lines. Call a game *cordial* if it satisfies this stronger form of niceness. Of course Wythoff Nim is cordial. With some careful work we expect one can obtain a specific bound on the distance of $P$-positions from the lines in terms of the constant appearing in the cordiality condition, but we have not checked this carefully. Empirically it seems that $(1,2)$GDWN is nice but not cordial. It seems as well that Conjecture 2 does not hold if one replaces niceness with cordiality; in general appending $(p,q)$ to $\mathcal{D}$ may downgrade cordial games to nice ones.

Additionally, this theorem holds not just for nice GDNs but in fact for an even broader class of nice variants of Wythoff Nim. For instance, define $\mathcal{D}\mathrm{GDN}_k + \mathcal{J}$, with $\mathcal{J}$ a finite set of vectors, to be $\mathcal{D}\mathrm{GDN}_k$ where additionally one may take $j_1$ stones from the first pile and $j_2$ stones from the second pile with $(j_1,j_2)\in\mathcal{J}$, and $k$ is now allowed to be infinite (so no move of the form $(k,k)$ is allowed unless that is an element of $\mathcal{J}$). This is an extremely broad class of variants, containing Maharaja Nim [LW12], Ryūō Nim [Miy+17], $Wyt_K$ [Duc+09] where $K$ is finite, and so on. If one proves that their variant of interest is nice (or cordial), they immediately know the asymptotic behavior of the $P$-positions. Furthermore, they know many $(p,q)$ such that appending $(p,q)$ to $\mathcal{D}$ changes the asymptotic behavior of the $P$-positions (and, if Conjecture 2 holds, *all* such $(p,q)$). For these variants we make the additional conjecture:

**Conjecture 3.** *If $\mathcal{D}GDN_k + \mathcal{J}$ is nice (cordial) for any finite $\mathcal{J}$ then $\mathcal{D}GDN_k + \mathcal{J}'$ is nice (cordial) for all finite sets $\mathcal{J}'$.*

Combined with Theorem 1, this is a vast generalization of Conjecture 4.1 in [LW12] that (in our notation) the $P$-positions of $\varnothing\mathrm{GDN}_1 + \{(j,\ell),(\ell,j)\}$ are bounded distance from the lines of slope $\phi$ and $1/\phi$, for any $j,\ell$. It is not hard to imagine further generalizations, say to when $\mathcal{J}$ is infinite but extremely sparse.

## 3.2 Computing Specific Slopes

The system of rational equations in part (3) of Theorem 1 implies that for nice $\mathcal{D}\mathrm{GDN}_k$, the $P$-positions lie essentially on some lines whose slopes are algebraic. Note that for most cases of interest it is not known whether or not $\mathcal{D}\mathrm{GDN}_k$ is nice, and we make no attempt to prove this for any specific cases aside from $(1,-1)$GDN in Section 5. The system from (3) can be solved, in the sense of finding minimal polynomials for these algebraic slopes, by multiplying to clear denominators and finding a Gröbner basis of the resulting system of multivariate polynomials. Before introducing that complexity, we will compute some easy cases by hand.

### 3.2.1 Wythoff Nim and GDN$_k$

We begin with Wythoff Nim. Then $D = \{(0,1),(1,0),(1,1)\}$, so $S = \{s_1, s_2\}$ with $s_1 < 1 < s_2$. We have the system

$$\begin{aligned} 1 &= \delta_1 + \delta_2 \\ 1 &= \frac{\delta_1}{s_1} + \frac{\delta_2}{s_2} \\ 1 &= \frac{\delta_1}{1 - s_1} \\ 1 &= \frac{\delta_2}{s_2 - 1} \end{aligned}$$

where $\delta_i := \delta_{s_i}$. Solve the last two equations for $\delta_1$ and $\delta_2$ and plug into the first two equations. We get

$$\begin{aligned} 1 &= 1 - s_1 + s_2 - 1 \\ 1 &= \frac{1 - s_1}{s_1} + \frac{s_2 - 1}{s_2}. \end{aligned}$$

From the first equation get $s_2 - 1 = s_1$; the second equation becomes

$$\begin{aligned} 1 &= \frac{1 - s_1}{s_1} + \frac{s_1}{s_1 + 1} \\ s_1^2 + s_1 &= 1 - s_1^2 + s_1^2 \\ s_1 &= 1/\phi \end{aligned}$$

so $s_2 = 1 + 1/\phi = \phi$, as expected.

Now consider GDN$_k$ for arbitrary $k \geq 1$. Then (it appears) $D = \{(0,1),(1,0),(k,k)\}$, so $S = \{s_1, s_2\}$ with $s_1 < 1 < s_2$ once again. We have the system

$$\begin{aligned} 1 &= \delta_1 + \delta_2 \\ 1 &= \frac{\delta_1}{s_1} + \frac{\delta_2}{s_2} \\ 1 &= \frac{\delta_1}{k - ks_1} \\ 1 &= \frac{\delta_2}{ks_2 - k}. \end{aligned}$$

Repeating the same steps as for Wythoff Nim, we get

$$ks_1^2 + s_1 - k = 0$$

i.e. $s_1 = \frac{-1+\sqrt{1+4k^2}}{2k}$. Then $s_2 = \frac{1+\sqrt{1+4k^2}}{2k}$. In [DG09], after translating to our notation, it is conjectured that the $P$-positions of GDN$_2$ with $y > x$ have $x$ a bounded distance from $\left\lfloor \frac{n(3+\sqrt{17})}{4} \right\rfloor$ for some $n$ (depending on $x$). This is basically the statement $1/\delta_2 = \frac{3+\sqrt{17}}{2}$, so to prove this conjecture it suffices to prove GDN$_2$ is cordial. Well, the conjecture may be obtained as a combination of results in [DG09] and [LW12], with no need to invoke Theorem 1:

**Theorem 2.** *The nth leftmost $P$-position $(x, y)$ of GDN$_2$ with $y > x$ has $x$ a bounded distance from $\left\lfloor \frac{n(3+\sqrt{17})}{4} \right\rfloor$.*

*Proof.* In [DG09] it is shown in the proof of Proposition 15 that if $(x_n, y_n)$ are the $P$-positions of GDN$_2$ with $x_n$ increasing and $y_n \geq x_n$ with the sole exception $(x_3, y_3) = (3, 2)$, then

$$\begin{aligned} y_{4t} - x_{4t} &= 2t \\ y_{4t+1} - x_{4t+1} &= 2t \\ y_{4t+2} - x_{4t+2} &= 2t + 1 \\ y_{4t+3} - x_{4t+3} &= 2t - 1. \end{aligned}$$

Furthermore, the sequences $X = 1, 2, x_4, x_5, x_6, \ldots$ and $Y = 2, 3, y_4, y_5, y_6, \ldots$ are complementary by definition of the sequences $P_n$ and $Q_n$ in [DG09], and they satisfy $Y_n - X_n = n/2 + O(1)$ by the above, and $X$ is increasing, so by the Central Lemma (Lemma 1.4) of [LW12], there are $\alpha$ and $\beta$ such that $X_n - \alpha n = O(1)$ and $Y_n - \alpha n = O(1)$. This is sufficient to prove the conjecture about the asymptotic $P$-positions of $\mathrm{GDN}_2$ since $\delta(1-\alpha) + \alpha = (\alpha - 1)\alpha$ ((10) in [LW12]) implies $\alpha = \frac{3+\sqrt{17}}{4}$ as desired. □

Do note that the proof above suffices as a proof of cordiality of $\mathrm{GDN}_2$, so we can discover the splitting vectors using part (4) of Theorem 1, for instance.

#### 3.2.2 GDWNs and (1, 2)GDN

Once more variables are introduced the complexity of solving this system grows dramatically. In the case of $(1, 2)$GDWN, which perhaps the next simplest symmetrical case, it is conjectured that the game is nice and

$$D = \{(0, 1), (1, 0), (1, 1), (1, 2), (2, 1)\}.$$

Then the system is

$$\begin{aligned}
1 &= \delta_1 + \delta_2 + \delta_3 + \delta_4 \\
1 &= \frac{\delta_1}{s_1} + \frac{\delta_2}{s_2} + \frac{\delta_3}{s_3} + \frac{\delta_4}{s_4} \\
1 &= \frac{\delta_1}{1 - 2s_1} \\
1 &= \frac{\delta_2}{2s_2 - 1} + \frac{\delta_3}{2s_3 - 1} + \frac{\delta_4}{2s_4 - 1} \\
1 &= \frac{\delta_1}{1 - s_1} + \frac{\delta_2}{1 - s_2} \\
1 &= \frac{\delta_3}{s_3 - 1} + \frac{\delta_4}{s_4 - 1} \\
1 &= \frac{\delta_1}{2 - s_1} + \frac{\delta_2}{2 - s_2} + \frac{\delta_3}{2 - s_3} \\
1 &= \frac{\delta_4}{s_4 - 2}.
\end{aligned}$$

Find $\delta_1 = 1 - 2s_1$, $\delta_2 = \frac{s_1(1-s_2)}{1-s_1}$, $\delta_4 = s_4 - 2$, $\delta_3 = \frac{s_3-1}{s_4-1}$ using the 3rd, 5th, 6th, and 8th equations. Plugging into the remaining equations gives

$$\begin{aligned}
1 &= 1 - 2s_1 + \frac{s_1(1 - s_2)}{1 - s_1} + \frac{s_3 - 1}{s_4 - 1} + s_4 - 2 \\
1 &= \frac{1 - 2s_1}{s_1} + \frac{s_1(1 - s_2)}{s_2(1 - s_1)} + \frac{s_3 - 1}{s_3(s_4 - 1)} + \frac{s_4 - 2}{s_4} \\
1 &= \frac{s_1(1 - s_2)}{(2s_2 - 1)(1 - s_1)} + \frac{s_3 - 1}{(2s_3 - 1)(s_4 - 1)} + \frac{s_4 - 2}{2s_4 - 1} \\
1 &= \frac{1 - 2s_1}{2 - s_1} + \frac{s_1(1 - s_2)}{(2 - s_2)(1 - s_1)} + \frac{s_3 - 1}{(2 - s_3)(s_4 - 1)}.
\end{aligned}$$

Clearing denominators and expanding everything gives

$$
\begin{aligned}
0 &= 2s_1^2s_4 - 2s_1^2 - s_1s_2s_4 + s_1s_2 - s_1s_3 - s_1s_4^2 + 2s_1s_4 + s_3 + s_4^2 - 3s_4 + 1 \\
0 &= s_1^2s_2s_3s_4^2 - 2s_1^2s_2s_3 + s_1^2s_2s_4 + s_1^2s_3s_4^2 - s_1^2s_3s_4 - 3s_1s_2s_3s_4^2 + 2s_1s_2s_3s_4 + 2s_1s_2s_3 - s_1s_2s_4 + s_2s_3s_4^2 \\
&\quad - s_2s_3s_4 \\
0 &= 2s_1s_2s_3s_4 - 4s_1s_2s_3 + s_1s_2s_4 + s_1s_2 + 2s_1s_3s_4^2 - 4s_1s_3s_4 + 3s_1s_3 - s_1s_4^2 + s_1s_4 - s_1 - 4s_2s_3s_4^2 \\
&\quad + 4s_2s_3s_4 + 2s_2s_3 + 2s_2s_4^2 - 4s_2s_4 + 2s_3s_4^2 - 2s_3s_4 - s_3 - s_4^2 + 2s_4 \\
0 &= -s_1^2s_2s_3 + s_1^2s_2 - s_1^2s_3s_4 + 3s_1^2s_3 + 2s_1^2s_4 - 4s_1^2 + 2s_1s_2s_3s_4 + s_1s_2s_3 - 4s_1s_2s_4 + s_1s_2 - 2s_1s_3s_4 \\
&\quad - 4s_1s_3 + 4s_1s_4 + 2s_1 - s_2s_3s_4 - s_2s_3 + 2s_2s_4 + 2s_3s_4 + 2s_3 - 4s_4.
\end{aligned}
$$

In principle one can now use Gröbner bases to find minimal polynomials for $s_i$, but this is computationally difficult, so we do some more simplification. We can "cheat" a bit by noting that the set of $P$-positions is symmetric across $y = x$, so $s_4 = 1/s_1$ and $s_3 = 1/s_2$. In fact, with these substitutions, the first equation and second equation become the same, and the third equation becomes $1/s_1$ times the fourth equation. We have

$$
\begin{aligned}
0 &= -2s_1^4s_2 + s_1^3s_2^2 + 2s_1^3s_2 - s_1^3 - s_1^2s_2^2 + 3s_1^2s_2 + s_1^2 - 4s_1s_2 + s_2 \\
0 &= s_1^3s_2^2 - 5s_1^3s_2 + 3s_1^3 + s_1^2s_2^2 + 5s_1^2s_2 - 5s_1^2 - 4s_1s_2^2 + 5s_1s_2 + 2s_2^2 - 5s_2 + 2.
\end{aligned}
$$

Both of these have a factor of $1 - s_1$. Dividing this out, we have

$$
\begin{aligned}
0 &= 2s_1^3s_2 - s_1^2s_2^2 + s_1^2 - 3s_1s_2 + s_2 \\
0 &= -s_1^2s_2^2 + 5s_1^2s_2 - 3s_1^2 - 2s_1s_2^2 + 2s_1 + 2s_2^2 - 5s_2 + 2.
\end{aligned}
$$

A CAS will now happily spit out the Gröbner basis

$$
\begin{aligned}
\{&71766s_1 + 498080s_2^9 - 2036496s_2^8 + 4371580s_2^7 - 14079408s_2^6 + 24172278s_2^5 \\
&\quad - 7663985s_2^4 - 24413440s_2^3 + 25315751s_2^2 - 4628498s_2 - 1607628, \\
&4s_2^{10} - 20s_2^9 + 50s_2^8 - 145s_2^7 + 297s_2^6 - 238s_2^5 - 141s_2^4 + 383s_2^3 - 222s_2^2 + 20s_2 + 12\}.
\end{aligned}
$$

The latter element factors as

$$(s_2 - 1)(s_2 + 1)\left(4s_2^8 - 20s_2^7 + 54s_2^6 - 165s_2^5 + 351s_2^4 - 403s_2^3 + 210s_2^2 - 20s_2 - 12\right),$$

so the minimal polynomial of $s_2$ is

$$4x^8 - 20x^7 + 54x^6 - 165x^5 + 351x^4 - 403x^3 + 210x^2 - 20x - 12,$$

so $s_2$ and $s_3 = 1/s_2$ have approximate values

$$s_2 \approx 0.676681656,$$

$$s_3 \approx 1.477799775.$$

Using the minimal polynomial for $s_2$ and one of the polynomial equations in $s_1$ and $s_2$ we can find (with the help of a CAS) the minimal polynomial for $s_1$, which turns out to be

$$12x^8 + 20x^7 - 25x^6 - 56x^5 + 24x^4 + 64x^3 - 12x^2 - 16x + 4,$$

so $s_1$ and $s_4 = 1/s_1$ have approximate values

$$s_1 \approx 0.444894166,$$

$$s_4 \approx 2.247725583.$$

In [Lar12] the slopes of the top two lines (i.e. $s_3$ and $s_4$) are estimated to be $1.478\ldots$ and $2.247\ldots$, so the constants we obtain match experiment. Again, proving that $(1,2)$GDWN is nice is left open. The work in [Lar14], where essentially the conclusion of part (4) of Theorem 1 is proven for $(1,2)$GDWN (i.e. $k=1$, $\mathcal{D}=\varnothing$, $\mathcal{E}=\{(1,2),(2,1)\}$) using different techniques, unfortunately does not appear to show that $(1,2)$GDWN is nice.

With significant computational effort one can repeat this process for $(a,b)$GDWN and find $s_2$ is a root of

$$\begin{aligned}
&x^8\left(2a^4b^2+2a^3b^3-5a^3b^2-a^2b^3+2a^2b^2\right)\\
&+x^7\left(-2a^4b^2-2a^4b-4a^3b^3+2a^3b^2+3a^3b-2a^2b^4+8a^2b^3-4a^2b^2-a^2b-ab^3+ab^2\right)\\
&+x^6\left(2a^5b-4a^4b^2-2a^3b^3+24a^3b^2+4a^3b+4a^2b^4-2a^2b^3-14a^2b^2-a^2b-2ab^4+2ab^3+ab^2\right)\\
&+x^5\left(-2a^5b-6a^4b^2-4a^4b-a^4-4a^3b^2-5a^3b+2a^2b^4-30a^2b^3+10a^2b^2+3a^2b-2ab^5+6ab^4\right.\\
&\qquad\left.+7ab^3-3ab^2-b^4\right)\\
&+x^4\left(2a^5b+a^5+18a^4b^2+a^4b+20a^3b^3-16a^3b^2-6a^3b+6a^2b^4-4a^2b^3+16a^2b^2+2a^2b+2ab^5\right.\\
&\qquad\left.+3ab^4-2ab^3-2ab^2-b^5\right)\\
&+x^3\left(-2a^5b-10a^4b^2+2a^4b+a^4-24a^3b^3+10a^3b^2+3a^3b-18a^2b^4+28a^2b^3-10a^2b^2-3a^2b\right.\\
&\qquad\left.-2ab^5-9ab^3+3ab^2+b^4\right)\\
&+x^2\left(-2a^4b+6a^3b^3+2a^3b^2+2a^3b+8a^2b^4-4a^2b^3-2a^2b^2-a^2b+2ab^5+ab^2\right)\\
&+x\left(2a^4b^2+4a^3b^3-4a^3b^2-a^3b+2a^2b^4-10a^2b^3+4a^2b^2+a^2b-2ab^4+3ab^3-ab^2\right)\\
&+\left(-2a^3b^3+a^3b^2-2a^2b^4+5a^2b^3-2a^2b^2\right).
\end{aligned}$$

The other element of the Gröbner basis is of the form $p_1(a,b,s_2)s_1+p_2(a,b,s_2)$ for some polynomials $p_1$ and $p_2$. If written out in full it is tens of thousands of characters long, so we omit it. But in principle one could use this to find a minimal polynomial $s_1$ as well, or one could find an alternative Gröbner basis which spits out the minimal polynomial of $s_1$ immediately. Note that since $s_1$ is in $\mathbb{Q}[s_2]$, it is degree 8 at most.

Let us compute one asymmetrical game, $(1,2)$GDN. The $P$-positions are shown in Figure 3.

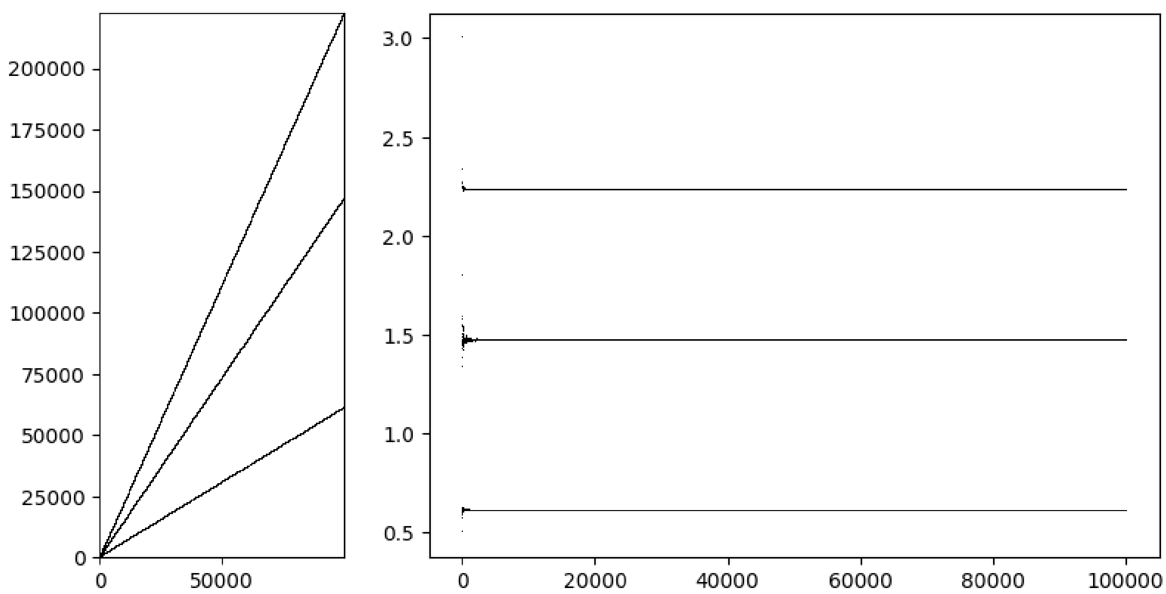


Figure 3: $(1,2)$GDN. On the left are the first 100000 $P$-positions. On the right, we plot $y/x$ against $x$ so the rapid convergence towards three lines of slopes approximately 0.61, 1.47, and 2.23 is clear.

The system is

$$
\begin{aligned}
1 &= \delta_1 + \delta_2 + \delta_3 \\
1 &= \frac{\delta_1}{s_1} + \frac{\delta_2}{s_2} + \frac{\delta_3}{s_3} \\
1 &= \frac{\delta_1}{1 - s_1} \\
1 &= \frac{\delta_2}{s_2 - 1} + \frac{\delta_3}{s_3 - 1} \\
1 &= \frac{\delta_1}{2 - s_1} + \frac{\delta_2}{2 - s_2} \\
1 &= \frac{\delta_3}{s_3 - 2}
\end{aligned}
$$

so $\delta_1 = 1 - s_1$, $\delta_3 = s_3 - 2$, $\delta_2 = s_1 - s_3 + 2$, leaving us with

$$
\begin{aligned}
1 &= \frac{1 - s_1}{s_1} + \frac{s_1 - s_3 + 2}{s_2} + \frac{s_3 - 2}{s_3} \\
1 &= \frac{s_1 - s_3 + 2}{s_2 - 1} + \frac{s_3 - 2}{s_3 - 1} \\
1 &= \frac{1 - s_1}{2 - s_1} + \frac{s_1 - s_3 + 2}{2 - s_2}
\end{aligned}
$$

Clearing denominators gives

$$
\begin{aligned}
0 &= s_1^2 s_3 - s_1 s_3^2 - s_1 s_2 s_3 + s_2 s_3 + 2 s_1 s_3 - 2 s_1 s_2 \\
0 &= s_1 s_3 - s_3^2 - s_1 - s_2 + 3 s_3 - 1 \\
0 &= s_1 s_3 - s_1^2 + s_2 - 2 s_3 + 2
\end{aligned}
$$

A Gröbner basis is

$$
\begin{aligned}
\{&8s_1 + s_3^5 - 2s_3^4 + s_3^3 - 7s_3 - 6, \\
&8s_2 - 3s_3^5 + 6s_3^4 + 5s_3^3 - 27s_3 + 10, \\
&s_3^6 - 3s_3^4 - 6s_3^3 + s_3^2 + 4s_3 + 4\},
\end{aligned}
$$

so the minimal polynomial of $s_1$, $s_2$, $s_3$ are

$$
\begin{aligned}
&x^6 + 3x^5 - 3x^4 - 12x^3 + 10x^2 - 8x + 4, \\
&x^6 + 3x^5 - 3x^4 - 20x^3 + 28x^2 - 12x + 4, \\
&x^6 - 3x^4 - 6x^3 + x^2 + 4x + 4,
\end{aligned}
$$

respectively, and they have approximate values

$$
\begin{aligned}
s_1 &\approx 0.610722699, \\
s_2 &\approx 1.469343290, \\
s_3 &\approx 2.228756687,
\end{aligned}
$$

matching experiment.

#### 3.2.3 Side Note: Detecting Nondominant Vectors Using the Rational System

One can compute that $(1, 3)$ is not a splitting vector of Wythoff Nim, so $(1, 3)$GDWN should have the same asymptotic behavior of $P$-positions as Wythoff Nim according to Conjecture 2. But what happens to the computation of $S$ and $\Delta$ if one does not check this and thinks that $(1,3)$ and $(3,1)$ are dominant in $(1,3)$GDWN? One will find the minimal polynomial of $s_2$ is

$$9x^8 - 54x^7 + 93x^6 - 296x^5 + 746x^4 - 1030x^3 + 597x^2 - 36x - 45$$

and the minimal polynomial of $s_1$ is

$$45x^8 + 66x^7 - 55x^6 - 162x^5 + 84x^4 + 270x^3 - 27x^2 - 54x + 9.$$

But then the approximate value of $s_1$ is $0.3846\ldots$, contradicting that $s_1 < 1/3$. In particular, this makes $\delta_1$ negative! So one will detect their mistake and conclude that either $(1,3)$GDWN is not nice or else it is nice and the dominant vectors are $\{(0,1),(1,0)\}$. Of course the belief is the latter possibility.

### 3.2.4 Further Extensions

In [DSS19], the following game is analyzed. Consider the points of $\mathbb{Z} \times \mathbb{Z}$ in a square spiral, beginning $(0,0),(1,0),(1,1),(0,1)$, and so on. If the point you are considering sees no previously placed queen, place a queen there. It was proven that the queens end up a constant distance from two lines through the origin of slopes $\psi$ and $-1/\psi$, where

$$\psi = \frac{1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}}{3}$$

is the *Tribonacci constant* with minimal polynomial $x^3 - x^2 - x - 1$. We can obtain this constant as the solution to a slightly modified system of equations. In particular, every row, column, diagonal, and antidiagonal of $\mathbb{Z} \times \mathbb{Z}$ ends up with a queen such that the blocks for each of them grows linearly up to a constant variation, so this game is cordial under an appropriate modification of that definition. This implies, in the spirit of Theorem 1, that the $P$-positions should lie a constant distance four rays based at the origin, with uniform density on these rays.

The system for computing the slopes of these rays is

$$\begin{aligned}
1 &= \delta_1 + \delta_4 \\
1 &= \delta_2 + \delta_3 \\
1 &= \frac{\delta_1}{s_1} + \frac{\delta_2}{-s_2} \\
1 &= \frac{\delta_3}{s_3} + \frac{\delta_4}{-s_4} \\
1 &= \frac{\delta_1}{s_1 - 1} + \frac{\delta_2}{-s_2 + 1} \\
1 &= \frac{\delta_3}{s_3 - 1} + \frac{\delta_4}{-s_4 + 1} \\
1 &= \frac{\delta_1}{s_1 + 1} + \frac{\delta_4}{s_4 + 1} \\
1 &= \frac{\delta_2}{s_2 + 1} + \frac{\delta_3}{s_3 + 1}.
\end{aligned}$$

Exploiting symmetry we find $s_1 = s_3$ and $s_2 = s_4$, and the same for the $\delta$'s. So the system is

$$\begin{aligned}
1 &= \delta_1 + \delta_2 \\
1 &= \frac{\delta_1}{s_1} + \frac{\delta_2}{-s_2} \\
1 &= \frac{\delta_1}{s_1 - 1} + \frac{\delta_2}{-s_2 + 1} \\
1 &= \frac{\delta_1}{s_1 + 1} + \frac{\delta_2}{s_2 + 1}.
\end{aligned}$$

From the first two equations find $\delta_1 = \frac{s_1(1+s_2)}{s_1+s_2}$ and $\delta_2 = \frac{s_2(1-s_1)}{s_1+s_2}$. Then clearing denominators in the last two equations gives

$$0 = s_1^2 + s_2^2 - 2s_1$$
$$0 = 2s_1^2 s_2 + s_1^2 + s_2^2.$$

A Gröbner basis is

$$\{s_1 - 4s_2^5 + 2s_2^4 - s_2^3 - 8s_2^2, s_2^6 + 2s_2^3 + s_2^2\}$$

so one finds the minimal polynomials of $s_1$ and $s_2$ are

$$x^3 - x^2 - x - 1 \qquad \text{and} \qquad x^3 - x^2 + x + 1,$$

respectively, and $s_1$ and $s_2$ are $\psi$ and $-1/\psi$ as expected. Actually, the system is simple enough that one does not even need to appeal to Gröbner bases to find these minimal polynomials.

The final example we look at is $k$-Wythoff Nim, first considered in [Hol68], in which the legal moves are to remove any number of stones from either pile or $a$ stones from one pile and $b$ from the other provided

$$|a - b| \leq k.$$

The case $k = 0$ is Wythoff Nim. It is no longer the case that every diagonal has a $P$-position, but instead now every $(k+1)$st diagonal has a $P$-position. So the appropriate system is

$$1 = \delta_1 + \delta_2$$
$$1 = \frac{\delta_1}{s_1} + \frac{\delta_2}{s_2}$$
$$\frac{1}{k+1} = \frac{\delta_1}{1 - s_1}$$
$$\frac{1}{k+1} = \frac{\delta_1}{s_2 - 1}.$$

Notice that this is the same system as for $\mathrm{GDN}_k$, but with $k$ replaced by $1/(k+1)$. Hence $s_1 = \frac{-k-1+\sqrt{k^2+2k+5}}{2}$ and $s_2 = \frac{k+1+\sqrt{k^2+2k+5}}{2}$. This is known to be correct, and in fact the $P$-positions are given by the Beatty sequence

$$\left(\left\lfloor \frac{1 - k + \sqrt{k^2 + 2k + 5}}{2} n \right\rfloor, \left\lfloor \frac{3 + k + \sqrt{k^2 + 2k + 5}}{2} n \right\rfloor\right)$$

for each $n \geq 0$, and the reflection of these points over $y = x$ [Hol68].

### 3.3 Notes on Conjecture 1

First let us prove the "only if" direction of Conjecture 1. This is basically a direct consequence of part (4) of Theorem 1 (along with the fact that Wythoff Nim is nice so that Theorem 1 applies).

**Theorem 3.** *If $(p, q)$ is a splitting pair then $\lim_{n\to\infty} \frac{y_n}{x_n}$ is not equal to $\phi$, where $\{(x_n, y_n)\}$ are the $P$-positions of $(p, q)$GDWN such that $x_1 \leq x_2 \leq \dots$ and $y_n > x_n$.*

*Proof.* Let $s_1 = 1/\phi$ and $s_2 = \phi$ be the slopes for Wythoff Nim (i.e. $\varnothing\mathrm{GDN}_1$) and $\delta_1 = 2 - \phi$ and $\delta_2 = \phi - 1$ the associated densities. If $(p, q)$ with say $q > p$ is a $P$-position of Wythoff Nim, then $p = \lfloor n\phi \rfloor$ and $q = \lfloor n\phi^2 \rfloor$ for some $n$. Then $q - p = n$ ($\phi^2 = \phi + 1$) so $q/p = 1 + n/\lfloor n\phi \rfloor > 1 + 1/\phi = \phi$. Let $k$ be the smallest integer so that $k/\phi > n$. Then $p = k-1$ and $q = k-1+n$, so

$$\begin{aligned}\sum_{\substack{s\in S\\ s<q/p}} \frac{\delta_s}{q-ps} &= \frac{\delta_1}{q-ps_1} + \frac{\delta_2}{q-ps_2}\\ &= \frac{\phi-1}{(k-1+n)\phi-(k-1)} + \frac{\phi-1}{(k-1+n)-(k-1)\phi}\\ &> \frac{\phi-1}{(k-1+k/\phi)\phi-(k-1)} + \frac{\phi-1}{(k-1+k/\phi)-(k-1)\phi}\\ &= \frac{\phi-1}{k\phi-(\phi-1)}+1\\ &> 1,\end{aligned}$$

so the $P$ positions of $(p,q)$GDWN are not $o(n)$ from lines of slopes $\phi$ and $1/\phi$.

If $(p-1,q-1)$ is a $P$-position of Wythoff Nim, then $p = \lfloor n\phi\rfloor + 1$ and $q = \lfloor n\phi^2\rfloor + 1$ for some $n$. Then $q/p < \phi$. Letting $k$ be the smallest integer so that $k/\phi > n$ as before, we have $p = k$ and $q = k+n$, so

$$\begin{aligned}\sum_{\substack{s\in S\\ s>q/p}} \frac{\delta_s}{ps-q} &= \frac{\delta_2}{ps_2-q}\\ &= \frac{\phi-1}{k\phi-(k+n)}\\ &> \frac{\phi-1}{k\phi-(k+(k-1)/\phi)}\\ &= 1,\end{aligned}$$

so the $P$ positions of $(p,q)$GDWN are not $o(n)$ from lines of slopes $\phi$ and $1/\phi$.

For $(p,q)$ with $q < p$, then $(q,p)$ is one of the moves added when going from Wythoff Nim to $(p,q)$GDWN, so the same analysis holds as above, just swapping $p$ and $q$. □

It is really just a coincidence due to simplicity of the characterization of $P$-positions of Wythoff Nim that the splitting vectors of Wythoff Nim are related so closely to the $P$-positions of Wythoff Nim. For instance, the splitting vectors of $\mathrm{GDN}_k$ (assuming this game is nice) may be calculated in a manner similar to the proof above to be

$$\left\{(n,m),(m,n) \mid m>n, |ns_k-m| < \frac{s_k}{s_k+1}\right\}$$

where $s_k = \frac{1+\sqrt{1+4k^2}}{2k}$. The condition that $(a,b)$ is splitting is basically that $b/a$ is extremely close to some $s \in S$ so that one of the terms in one of the possible rational expressions is large, which is similar to the characterization that the $P$-positions are close to lying on lines whose slopes are in $S$, though in general the splitting vectors appear much closer to these lines than the $P$-positions and they are more numerous. See Figure 4 for a direct comparison in a few cases. Are there other games so that the $P$-positions and splitting vectors are closely related?

# 4 Cruel Games and Intransitive Dice

Say a variant is *cruel* if it is not nice. Are there cruel GDNs? Although none are proven, it appears there are many examples, the simplest of which is probably $(3,5)$GDWN, shown in Figure 5. If $(3,5)$GDWN were nice, the slopes of the lines that the $P$-positions lie $o(n)$ distance from would be roughly 0.56813, 0.65021, 1.53796, and 1.76014, but the $P$-positions appear to be $\Omega(n)$ distance from these lines (though they still lie fairly close to them). See the Appendix in [Lar12] for many other interesting examples of suspected cruel games.

We make some conjectures for how one might analyze these games. Our ideas are loosely similar to the main idea in [Sim21], namely to consider not the specific placements of $P$-positions but instead a distribution

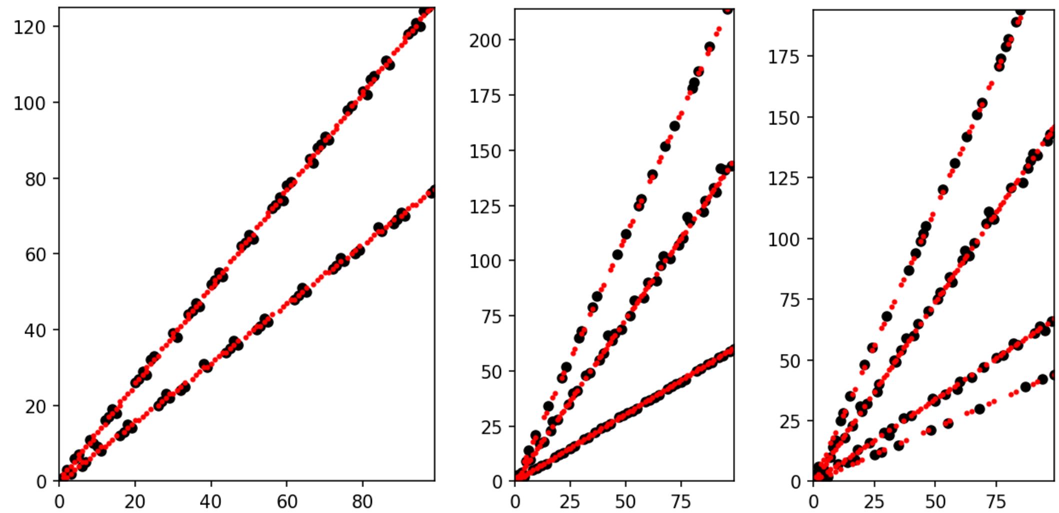


Figure 4: The $P$-positions (large black dots) and splitting vectors (small red dots) for $\mathrm{GDN}_2$ (left), $(1,2)$GDN (middle) and $(1,2)$GDWN (right).

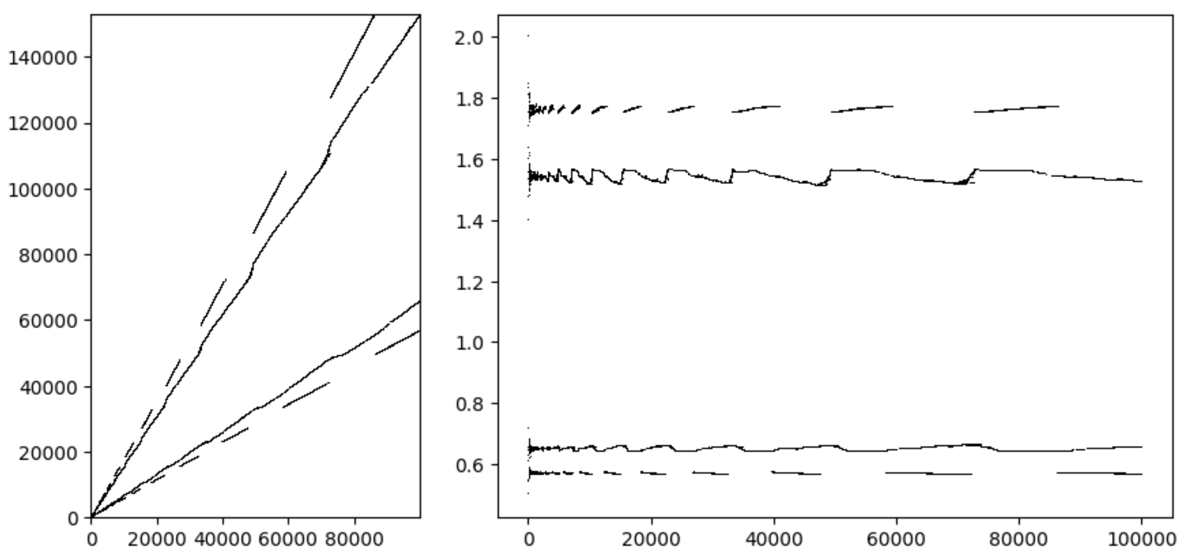


Figure 5: $(3,5)$GDWN. On the left are the first 100000 $P$-positions. On the right, we plot $y/x$ against $x$. Note the apparent quasiperiodicity and lack of convergence, unlike in Figure 3 for example. We estimate the quasiperiod $\sigma$ to be roughly 0.68.

of placements of $P$-positions (in [Sim21], "$P$-positions" are instead queen placements on an $n \times n$ chessboard, but there is the connection due to Section 2).

### 4.1 First Pass

Instead of a finite set of slopes $S$ and densities $\Delta$, let $\mu$ be a Borel measure on $[0,\infty)$ so that the number of $P$-positions $(x,y)$ with $s_1 < |y/x| < s_2$ is asymptotically $\mu((s_1,s_2))$. Recall from the proof of part (4) of Theorem 1 that we had

$$\sum_{\substack{s\in S\\ s>d/c}} \frac{\delta_s}{cs-d} \leq 1 \qquad \text{and} \qquad \sum_{\substack{s\in S\\ s<d/c}} \frac{\delta_s}{d-cs} \leq 1$$

for all $(c,d) \in \tilde{\mathcal{D}}$. Then we can view determining $S$ and $\Delta$ as a kind of optimization problem, to find the "lowest" choice of $S$ and $\Delta$ such that these inequalities are satisfied. Intuitively, lowest should mean roughly that if one draws an $s \in S$ at random with weight $\delta_s$, the chosen number will be usually be smaller than for any other $S$ and $\Delta$ satisfying the inequalities. That is, we wish for

$$\sum_{s_i \in S} \sum_{s_j \in S'} \operatorname{sgn}(s_i - s_j) \delta_{s_i} \delta'_{s_j} > 0$$

whenever $(S', \Delta')$ is another possible set of slopes and densities satisfies the necessary inequalities. For instance, the $(S, \Delta)$ for $(1,2)$GDWN computed in Section 3.2 satisfies the inequalities for Wythoff Nim (they are a subset of the inequalities for $(1,2)$GDWN), but one can compute that the solution for Wythoff Nim "beats" the solution for $(1,2)$GDWN at about 50.7% of columns (corresponding to a value $\approx 0.014$ in the double sum above).

The appropriate condition on $\mu$ is then to demand that

$$\int_{d/c}^{\infty} \frac{1}{cs - d} \, \mathrm{d}M(s) \leq 1 \qquad \text{and} \qquad \int_{0}^{d/c} \frac{1}{d - ac} \, \mathrm{d}M(s) \leq 1$$

for all $(c,d) \in \tilde{\mathcal{D}}$, where $M$ is such that $\mu$ is the Lebesgue-Stieltjes measure of $M$. For simplicity we will ignore the complications due to measure theory from now on and just write

$$\int_{d/c}^{\infty} \frac{\mu(s)}{cs - d} \, \mathrm{d}s \leq 1 \qquad \text{and} \qquad \int_{0}^{d/c} \frac{\mu(s)}{d - ac} \, \mathrm{d}s \leq 1$$

instead. Call a $\mu$ *valid* if it satisfies these inequalities. We should prefer $\mu$ to $\mu'$ if

$$\int_0^{\infty} \int_0^{\infty} \operatorname{sgn}(s_1 - s_2) \mu(s_1) \mu'(s_2) \, \mathrm{d}s_2, s_1 > 0.$$

Unfortunately (perhaps), the preference relation we have defined is not transitive! Indeed, it is exactly the reverse of the relation that gives rise to intransitive dice. However, there is actually a (weak) maximum value under this relation:

**Theorem 4.** *There is a valid $\mu^*$ such that for all other valid $\mu$,*

$$\int_0^{\infty} \int_0^{\infty} \operatorname{sgn}(s_1 - s_2) \mu^*(s_1) \mu(s_2) \, \mathrm{d}s_2, s_1 \geq 0.$$

*Proof.* First we note in any $\mathcal{D}\mathrm{GDN}_k$ that there is an upper bound for $y/x$ where $(x,y)$ is a $P$-position. Notice that each $P$-position to the left of $(x,y)$ can eliminate at most $|\mathcal{D}| + 2$ possible positions from the $x$th column (one for each $(a,b) \in \mathcal{D}$, plus one for $(k,k)$ moves and one for $(1,0)$ moves), so $y \leq (|\mathcal{D}| + 2)x + 1$. Thus instead of considering measures on $[0, \infty)$ we can consider measures on $[0, |\mathcal{D}| + 3]$, say.

Then consider the following game: Alice and Bob simultaneously come up with valid $\mu_A$ and $\mu_B$, then interpret their $\mu$'s as a probability measures on $[0, |\mathcal{D}| + 3]$ and draw $s_A$ and $s_B$ according to this measure. Then the winner is whoever drew the smaller number. Note that in the appropriate topology the set of valid $\mu$ is compact; indeed, the set of $\mu$ satisfying just one of the necessary inequalities is compact (this is, however, why we needed to restrict to $[0, |\mathcal{D}| + 3]$), so their intersection, that is the set of valid $\mu$, is compact. Thus there is a mixed-strategy Nash equilibrium for this game, which is a probability measure $X$ on the space of valid $\mu$. Using this we will construct a pure-strategy Nash equilibrium, which is our $\mu^*$.

Note that the set of $\mu$ satisfying just one of the necessary inequalities is also convex, which follows by linearity of the integral and the fact that $1/(as - b)$ and $1/(b - as)$ are monotone in $s$ on the relevant ranges. Thus the set of valid $\mu$ is convex, so $\mu^*$ given by

$$\mu^*(U) = \int_{\text{valid } \mu} \mu(U) \, \mathrm{d}Y(\mu)$$

is valid, where $Y$ is such that $X$ is the Lebesgue-Stieltjes measure of $Y$. The strategy "Always choose $\mu^*$" at least ties with $X$ since the probability the number chosen by both strategies lies in $U \subset [0, |\mathcal{D}| + 3]$ is equal. Thus $\mu^*$ is a pure-strategy equilibrium for this game, which implies the theorem statement. ☐

## 4.2 Second Pass

Unfortunately, in cruel games the $P$-positions appear that they will not approach this magical $\mu^*$. In fact they will not approach any measure, for they exhibit some kind of quasiperiodicity. In particular, it looks like one can partition $[0,\infty)\cap\mathbb{Z}$ into contiguous sets $X_i$ such that $|X_{i+1}|\geq r|X_i|$ for some $r>1$ such that there is a $U$ and $\alpha<\beta$ such that the lim inf as $t\to\infty$ of the number of $P$-positions with $y/x\in U$ and $x\in X_{2t}$ is at least $\beta$, but the lim sup of the number of $P$-positions with $y/x\in U$ and $x\in X_{2t+1}$ is at most $\alpha$.

Our second pass remedies this deficiency. Consider $\mu$ a measure on $[0,1]\times[0,\infty)$, with the following properties:

- $\mu((a,b)\times[0,\infty))=b-a$ for $0\leq a\leq b\leq 1$.
- For each $(c,d)\in\tilde{\mathcal{D}}$ other than $(0,1)$, $\mu(\{(x,dx/c)\mid a\leq x\leq b\})\leq c(b-a)$ for all $a,b$.

Call $\mu$ *quasiperiodic* if there is a $0<\sigma<1$ such that $\mu(U)=\sigma\mu(\sigma U)$ for all $U$, where $\sigma U=\{(\sigma x,\sigma y)\mid (x,y)\in U\}$. For quasiperiodic $\mu,\mu'$ say $\mu\sim\mu'$ if there is are $\tau,\tau'$ such that $\mu(U)=\tau\mu'(\tau U)$ and $\mu'(U)=\tau'\mu(\tau' U)$ for all $U$. One can verify this is an equivalence relation. Denote by $\mathcal{F}(\mu)$ the equivalence class of $\mu$ under $\sim$. Notice for any $\mu$ that $\mathcal{F}(\mu)$ is compact. Indeed, one can define a homeomorphism (in the suitable topology) from the elements of $\mathcal{F}(\mu)$ to a circle; the $\mu'$ at angle $\theta$ has $\tau=\sigma^{\theta/(2\pi)}$ relating it to $\mu$ at angle 0. Then we make the following (imprecise) conjecture:

**Conjecture 4.** *For any $\mathcal{D}GDN_k$, there exists an equivalence class $\mathcal{F}$ such that the $P$-positions with $x$-coordinate less than $N$ approach $\mathcal{F}$ as $N\to\infty$.*

We clarify the meaning of "approach." Note that the $P$-positions with $x$-coordinate less than $N$ give a $\mu_N$ by placing weight $N$ at the square with side length $1/N$ and upper-right corner $(x/N,y/N)$, with $(x,y)$ a $P$-position. One possible version of this conjecture is that for any $U$, the infimum of $|\mu_N(U)-\mu'(U)|$ for $\mu'\in\mathcal{F}$ approaches 0 as $N\to\infty$.

Intuitively, it is precisely the intransitivity of the preference relation that leads to quasiperiodicity. In particular, we suspect for the $\mathcal{F}$ in the conjecture statement that for sufficiently small $\varepsilon$, the element of $\mathcal{F}$ at angle $\theta+\varepsilon$ is preferred to the element at angle $\theta$, where preference is defined by looking at the restriction of the $\mu$'s to $\{1\}\times[0,\infty)$. In other words, the elements of $\mathcal{F}$ form a continuous cycle of intransitive dice.

# 5 The Case of Greedy Queens

In [DSS19] it is conjectured that the $P$-positions of $(1,-1)$GDN lie at a bounded distance from lines of slopes $\phi$ and $1/\phi$, the same as in Wythoff Nim. It suffices to prove that $(1,-1)$GDN is cordial. We sketch the first half of a proof of this. The technique is a variation of the string substitution technique used in [LW12].

There are four sets of lines that are important in the analysis of $(1,-1)$GDN, namely the rows $R_y=\{(x,y)\mid x\in\mathbb{Z}\}$ for each $y\geq 0$, the columns $C_x=\{(x,y)\mid y\in\mathbb{Z}\}$ for each $x\geq 0$, the diagonals $D_k=\{(z,k+z)\mid z\in\mathbb{Z}\}$ for $k\in\mathbb{Z}$, and the antidiagonals $A_k=\{(w,k-w)\mid w\in\mathbb{Z}\}$ for $k\geq 0$.

Realize that if a $P$-position is placed above the line $y=x$, then its position is entirely determined by the last $P$-position above the line $y=x$. In particular, if that $P$-position was $(a,b)$, the new $P$-position will be $(a+k,b+k+1)$ for some $k\geq 1$. This is because (inductively) all diagonals $D_k$ from $k=0$ to $b-a$ already have a $P$-position, so the lowest available location for the new $P$-position is $(a+k,b+k+1)$, and (inductively) this sees no other $P$-positions. Thus the "hard part" in determining the $P$-position placement is determining:

(1) when a $P$-position lies above or below the line $y=x$, and

(2) if it lies below $y=x$, where it is placed.

The key is then to try to find the smallest amount of information that can determine these two things at any point in time, and determining how that information changes after a $P$-position is placed. First, call a $P$-position *near* if it lies below $y=x$ and *far* otherwise. Then define the following:

- $n$: The current column being considered.
- $r$: The smallest $y$ for which $R_y$ is not covered by a $P$-position.
- $d$: The smallest $k$ for which $D_k$ is not covered by a $P$-position (this is negative in general).
- $u$: The largest $k$ for which $D_k$ is not covered by a $P$-position.
- $r_f$: From $y = r$ to $n + d$, the rows $R_y$ that are covered by a far $P$-position. This set is called the *far row pattern.*
- $r_n$: From $y = r$ to $\infty$, the $R_y$ covered by a near $P$-position. This is the *near row pattern.*
- $d_n$: From $k = -\infty$ to $d$, the $D_k$ covered by a (near) $P$-position. This is the *diagonal pattern.*
- $a_f$: From $k = n + r$ to $2n + d$, the $A_k$ covered by a far $P$-position. This is the *far antidiagonal pattern.*
- $a_n$: From $k = n + r$ to $\infty$, the $A_k$ covered by a near $P$-position. This is the *near antidiagonal pattern.*

We regard these as functions of $n$, so we may write $r(n)$, $r_f(n)$, etc.

First we prove:

**Lemma 1.** *$(n, r, d, u, r_f, r_n, d_n, a_f, a_n)$ are sufficient to determine the next $P$-position.*

*Proof.* From the definition of $r$, all of the positions $(n, 0)$ through $(n, r - 1)$ see a $P$-position along the same row. From $d$ and $u$, all the positions $(n, n + d + 1)$ through $(n, n + u - 1)$ see a $P$-position along the same diagonal. By an easy inductive argument, $(n, n + u)$ sees no $P$-position. Thus the $P$-position in column $n$ is either one of the positions $(n, n+r)$ through $(n, n+d)$ or it is $(n, n+u)$. Then clearly $(r_f, r_n, d_n, a_f, a_n)$ are sufficient to determine which (if any) of the positions $(n, n + r)$ through $(n, n + d)$ see a $P$-position, noting for the case of diagonals that no far $P$-position lies on the same diagonal as any of the positions $(n, n + r)$ through $(n, n + d)$. The new $P$-position is placed at the lowest of these positions that do not see an earlier $P$-position, or $(n, n + u)$ if all of them see an earlier $P$-position. □

The near and far patterns are separated because, for example, the pair $(r_f, r_n)$ behaves more predictably than $r_f \cup r_n$ after a $P$-position is determined, since the near patterns are determined by the recent near $P$-positions while the far patterns are determined by the far $P$-positions from long ago.

We are really more interested in the location of the next $P$-position relative to $n$, $r$, and $u$, rather than its absolute location. So we define:

- $w = n + d - r$, the *window size.*
- $r'_f = \{y - r \mid y \in r_f\}$,
- etc. (Each of the other patterns just get appopriately shifted by multiples of $n$ and $r$.)

Call $(w, r'_f, r'_n, d'_n, a'_f, a'_n)$ the *reduced information tuple*, or just *tuple*. Along with $n$, $r$, and $u$, the tuple determines the $P$-position in column $n$. But after this placement, what is the next tuple? This is not determined in general, since as $n$ increments, $w$ increments (up to changes in $d$ and $r$), so the window determining $r_f$ grows by one and the window for $a_f$ grows by two. Whether or not the new possible rows and antidiagonals are covered by far $P$-positions depends on the $P$-position placement from a long time ago. However, there are only a few possibilities in general, since adjacent far $P$-positions lie two rows or three antidiagonals apart at minimum, and (conjecturally) some constant distance apart at maximum, so for instance if $w \in r'_f(n)$ then certainly $w + 1 \notin r'_f(n + 1)$.

Thus we form a (infinite) directed graph whose vertices are the possible tuples and edges the possible transitions between them; what we have remarked is that the maximum outdegree is not too large. Label the edges with $(R, A)$, where $R$ and $A$ are binary strings that describe the far $P$-positions that are necessary in order for this edge to be the transition. For instance, the edge

$$(1, \varnothing, \varnothing, \varnothing, \{1\}, \varnothing) \to (2, \{2\}, \varnothing, \varnothing, \varnothing, \varnothing)$$

gets label $(1, 00)$.

Choose your favorite tuple (a good choice is the tuple that occurs most often empirically, which is $(0, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing))$, and for each closed walk based at that tuple, create a rule of the form $(R, A) \to (R', A')$, where $R$ and $A$ are the concatenation of the $R$ and $A$ for each edge in the walk and $R'$ and $A'$ describe the new $P$-position placements. If a vertex on the walk leads to a $P$-position being placed below $y = x$, then append 0 to $R'$ and 00 to $A'$. If it leads to a $P$-position being placed above $y = x$, then append 10 to $R'$ and 100 to $A'$.

Then the process for computing the $P$-positions (actually just whether each $P$-position is near or far, but that will be good enough) is that one starts with a pair of strings describing the $P$-positions from some point onwards in terms of rows and antidiagonals, then read both strings and find the earliest rule (in some order) that matches their prefixes. Pop those prefixes off of the front and add $R'$ and $A'$ to the end. If only finitely many rules are ever used, we can conclude that $(1, -1)$GDN is cordial.

In the Appendix we provide what we believe to be the minimal subgraph and set of rules. We provide the count of the number of times each vertex is visited and each rule is used from the $P$-positions in the first $10^7$ columns. Proving that no other tuples and rules appear appears very difficult, or at the very least extremely tedious. In [LW12], a similar string rewriting system is found for Maharaja Nim (GDN+$(1, 2)(2, 1)$), amounting to 14 rules, though it is suspected only 9 rules are actually ever used. The longest rule read a prefix of length 9. Their proof that this ruleset is sufficient eliminated 11 substrings from ever appearing and is the better part of a page of reasoning. In our case we have 73 rules that definitely appear, the longest of which reads 16 characters from one string and 42 from the other, so we suspect a proof similar to the one in [LW12] would take several pages at minimum, but it could end up much longer and harder due to complications arising from operating on two strings simultaneously and due to the increase in the maximum rule length.

In principle, we expect a similar string-rewriting method would produce a proof of cordiality for all cordial games. We call such a method a *multistring rewriting system*, with a *multistring* being a tuple of strings. This consists of a starting multistring $\sigma$ along with a sequence of rules (aka *dictionary*) $D$, where at each step one looks along the sequence until one matches the prefixes of every string in the current multistring, then pops off those prefixes and appends the associated suffixes. Each rule also determines the placement of some $P$-positions. In other words:

**Conjecture 5.** *For all cordial games, there is a finite multistring rewriting system that determines the $P$-position placement.*

Also, the number of strings being operated on should equal to the number of infinite families of moves minus 1. For instance in Maharaja Nim there are two infinite families of moves, namely removing any number of stones from the first pile or any number from the second pile, and in $(1, -1)$GDN there is a third, adding the ability to remove any number of stones from the first pile and add the same number to the second.

# 6 Concluding Remarks

One may regard parts (3) and (4) of Theorem 1 as sophisticated density arguments. We have proven one (admittedly strong) condition on a game in order for these density arguments to apply. We demonstrate how one can solve this system of equations to determine minimal polynomials for the (algebraic) slopes of lines that the $P$-positions lie near. In cases with more lines it may be impractical to find these minimal polynomials, but numerical solutions can also be found. Using this, one can come up with an endless number of precise conjectures about the asymptotic behavior of a wide variety of variants of Wythoff Nim.

As for cases where Theorem 1 does not seem to apply, we describe a potential avenue for analysis by making the problem continuous in a similar vein to [Sim21]. However, finding (or even approximating) a solution seems extremely difficult.

As for proving the cordiality of certain games, we describe an advanced string rewriting technique in a similar vein to [LW12]. The key insight is to operate on multiple strings simultaneously, though this makes proving that the multistring rewriting system covers all cases more difficult. We suspect for all cordial games that a similar technique could be used to prove cordiality, though much more research is required into this area.

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# A Greedy Queens Digraph and Multistring Rewriting

As described in Section 5, one can construct a directed graph whose vertices are possible tuples and whose edges represent transitions that can occur based on far $P$-position placements from many columns earlier. Then one can construct a multistring rewriting system by looking at closed walks in this digraph based at a particular tuple. In general this graph and multistring rewriting system are infinite, but it is conjectured that only finitely many vertices in the graph and rewriting rules ever occur in practice, which would imply the cordiality of $(1, -1)$GDN.

The digraph is given in Tables A1 and A2. Notice that many vertices have outdegree 1, indicating that the window did not grow (enough) in order for new far $P$-positions to affect the tuple.

The multistring rewriting system is given in Table A3. The interpretation is that a 1 is a $P$-position above the line $y = x$ and a 0 is a $P$-position on or below $y = x$. A possible seed multistring is

$$(01010001010101, 010010000001001001001000000100001000),$$

where at each stage, one constructs the new multistring by repeatedly matching prefixes according to the table of rules, appending the suffixes onto the multistring $(0, 0)$, until reaching the end of the current string. For instance, the seed multistring matches prefixes

$$(0101, 0100100000), (000101, 0100100100100000), (0101, 0100001000)$$

in order, meaning the next multistring is

$$(0101000101010100010010010101010001010,$$
$$01001000000100100100100000010000100001001001001000000100100).$$

One just needs that the seed multistring is long enough that the multistring grows enough to match more prefixes after one step.

Notice that the second prefix is irrelevant to determining the suffixes in all cases except when the first prefix is 00010101. This fact could potentially be used to greatly simplify a proof that this multistring rewriting system covers all possible cases.

| Vertex | $w$ | $r'_f$ | $r'_n$ | $d'_n$ | $a'_f$ | $a'_n$ | Outneighbors | Count |
|---|---|---|---|---|---|---|---|---|
| 1 | $0$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | 2, 4 | 1571734 |
| 2 | $1$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | 1, 3, 5, 19 | 1571733 |
| 3 | $-1$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | 1 | 1412793 |
| 4 | $1$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\{0\}$ | $\varnothing$ | 6, 7 | 978874 |
| 5 | $-2$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | 3 | 629763 |
| 6 | $2$ | $\varnothing$ | $\varnothing$ | $\{-1\}$ | $\varnothing$ | $\varnothing$ | 8, 13, 16 | 505678 |
| 7 | $2$ | $\varnothing$ | $\varnothing$ | $\{-2\}$ | $\varnothing$ | $\varnothing$ | 2 | 473195 |
| 8 | $2$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\varnothing$ | $\{1\}$ | 11, 18 | 287309 |
| 9 | $0$ | $\varnothing$ | $\varnothing$ | $\{-2\}$ | $\varnothing$ | $\varnothing$ | 2 | 243356 |
| 10 | $2$ | $\varnothing$ | $\varnothing$ | $\{-3\}$ | $\varnothing$ | $\varnothing$ | 9 | 189179 |
| 11 | $1$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\varnothing$ | $\{0\}$ | 10, 27, 33, 34 | 186808 |
| 12 | $3$ | $\varnothing$ | $\{2\}$ | $\varnothing$ | $\varnothing$ | $\{2\}$ | 2, 11, 24 | 184269 |
| 13 | $2$ | $\{1\}$ | $\varnothing$ | $\{0\}$ | $\{0\}$ | $\varnothing$ | 14, 15 | 170648 |
| 14 | $3$ | $\varnothing$ | $\varnothing$ | $\{-2,-1\}$ | $\varnothing$ | $\varnothing$ | 8, 17, 20, 29 | 160700 |
| 15 | $3$ | $\varnothing$ | $\varnothing$ | $\{-2,-1\}$ | $\{0\}$ | $\varnothing$ | 8, 17 | 160011 |
| 16 | $2$ | $\{1\}$ | $\varnothing$ | $\{0\}$ | $\varnothing$ | $\varnothing$ | 14 | 150063 |
| 17 | $2$ | $\{1\}$ | $\varnothing$ | $\{-1,0\}$ | $\varnothing$ | $\varnothing$ | 12, 21 | 125978 |
| 18 | $0$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\varnothing$ | $\{0\}$ | 2 | 102342 |
| 19 | $-3$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | 5 | 87481 |
| 20 | $2$ | $\{1\}$ | $\varnothing$ | $\{-1,0\}$ | $\{1\}$ | $\varnothing$ | 12 | 62031 |
| 21 | $3$ | $\{1\}$ | $\varnothing$ | $\{0,1\}$ | $\{2\}$ | $\varnothing$ | 23, 49 | 59625 |
| 22 | $1$ | $\varnothing$ | $\{2\}$ | $\varnothing$ | $\varnothing$ | $\{0\}$ | 10, 39, 43 | 56962 |
| 23 | $4$ | $\{1\}$ | $\varnothing$ | $\{-2,-1,0\}$ | $\varnothing$ | $\varnothing$ | 12 | 55636 |
| 24 | $0$ | $\varnothing$ | $\{2\}$ | $\varnothing$ | $\varnothing$ | $\{1\}$ | 22 | 55636 |
| 25 | $-1$ | $\varnothing$ | $\varnothing$ | $\{-3\}$ | $\varnothing$ | $\varnothing$ | 9 | 54003 |
| 26 | $1$ | $\varnothing$ | $\varnothing$ | $\{-4\}$ | $\varnothing$ | $\varnothing$ | 25 | 40345 |
| 27 | $2$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\{0\}$ | $\varnothing$ | 28 | 37766 |
| 28 | $3$ | $\varnothing$ | $\varnothing$ | $\{-3\}$ | $\varnothing$ | $\varnothing$ | 26, 65, 69 | 37766 |
| 29 | $2$ | $\varnothing$ | $\varnothing$ | $\{-1,0\}$ | $\{1\}$ | $\varnothing$ | 12, 30 | 30360 |
| 30 | $3$ | $\{2\}$ | $\varnothing$ | $\{0,1\}$ | $\{0\}$ | $\varnothing$ | 32, 37 | 30111 |
| 31 | $4$ | $\varnothing$ | $\{3\}$ | $\varnothing$ | $\varnothing$ | $\{3\}$ | 11, 35, 41 | 20506 |
| 32 | $4$ | $\{1\}$ | $\varnothing$ | $\{-1,0,1\}$ | $\varnothing$ | $\varnothing$ | 31, 36 | 16219 |
| 33 | $0$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | 2 | 15707 |
| 34 | $-1$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | 33 | 15458 |
| 35 | $0$ | $\varnothing$ | $\{3\}$ | $\varnothing$ | $\varnothing$ | $\{2\}$ | 22, 38 | 15192 |
| 36 | $4$ | $\{1,3\}$ | $\varnothing$ | $\{0,1,2\}$ | $\{2\}$ | $\varnothing$ | 42, 62 | 14627 |
| 37 | $4$ | $\{1\}$ | $\varnothing$ | $\{-1,0,1\}$ | $\{1\}$ | $\varnothing$ | 45, 48 | 13892 |
| 38 | $1$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | 34 | 13866 |
| 39 | $2$ | $\varnothing$ | $\varnothing$ | $\{-4\}$ | $\varnothing$ | $\varnothing$ | 25 | 13658 |
| 40 | $5$ | $\varnothing$ | $\{4\}$ | $\varnothing$ | $\varnothing$ | $\{4\}$ | 41, 53 | 13594 |
| 41 | $1$ | $\varnothing$ | $\{2\}$ | $\varnothing$ | $\varnothing$ | $\{1\}$ | 44, 61 | 13508 |
| 42 | $5$ | $\{1\}$ | $\varnothing$ | $\{-2,-1,0,1\}$ | $\{2\}$ | $\varnothing$ | 31, 47 | 12840 |
| 43 | $0$ | $\varnothing$ | $\{2\}$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | 2 | 11112 |
| 44 | $-1$ | $\varnothing$ | $\{2\}$ | $\varnothing$ | $\varnothing$ | $\{0\}$ | 43 | 9786 |
| 45 | $4$ | $\{1,3\}$ | $\varnothing$ | $\{0,1,2\}$ | $\{0,3\}$ | $\varnothing$ | 46 | 8758 |
| 46 | $5$ | $\{1\}$ | $\varnothing$ | $\{-2,-1,0,1\}$ | $\{0\}$ | $\varnothing$ | 31 | 8758 |
| 47 | $4$ | $\{1,3\}$ | $\varnothing$ | $\{-1,0,1,2\}$ | $\{1\}$ | $\varnothing$ | 40 | 7732 |
| 48 | $4$ | $\{1\}$ | $\varnothing$ | $\{0,1,2\}$ | $\{0,3\}$ | $\varnothing$ | 51, 64 | 5134 |
| 49 | $4$ | $\varnothing$ | $\varnothing$ | $\{-2,-1,0\}$ | $\varnothing$ | $\varnothing$ | 50 | 3989 |
| 50 | $3$ | $\{2\}$ | $\varnothing$ | $\{-1,0,1\}$ | $\{1\}$ | $\varnothing$ | 31, 67 | 3989 |

Table A1: Tuple digraph for $(1,-1)$GDN, part 1. Part 2 is Table A2.

| Vertex | $w$ | $r'_f$ | $r'_n$ | $d'_n$ | $a'_f$ | $a'_n$ | Outneighbors | Count |
|---|---|---|---|---|---|---|---|---|
| 51 | 5 | $\varnothing$ | $\varnothing$ | $\{-2,-1,0,1\}$ | $\{0\}$ | $\varnothing$ | 52 | 3808 |
| 52 | 4 | $\{3\}$ | $\varnothing$ | $\{-1,0,1,2\}$ | $\varnothing$ | $\varnothing$ | 40 | 3808 |
| 53 | 1 | $\varnothing$ | $\{3\}$ | $\varnothing$ | $\varnothing$ | $\{2\}$ | 54 | 3808 |
| 54 | 0 | $\varnothing$ | $\{3\}$ | $\varnothing$ | $\varnothing$ | $\{1\}$ | 55 | 3808 |
| 55 | 1 | $\varnothing$ | $\{3\}$ | $\varnothing$ | $\varnothing$ | $\{0\}$ | 56 | 3808 |
| 56 | 2 | $\varnothing$ | $\{1,3\}$ | $\varnothing$ | $\{0\}$ | $\{1\}$ | 57 | 3808 |
| 57 | 2 | $\varnothing$ | $\{1,3\}$ | $\varnothing$ | $\varnothing$ | $\{0\}$ | 58 | 3808 |
| 58 | 3 | $\varnothing$ | $\varnothing$ | $\{-5\}$ | $\varnothing$ | $\varnothing$ | 59 | 3808 |
| 59 | $-1$ | $\varnothing$ | $\varnothing$ | $\{-4\}$ | $\varnothing$ | $\varnothing$ | 60 | 3808 |
| 60 | 0 | $\varnothing$ | $\varnothing$ | $\{-3\}$ | $\varnothing$ | $\varnothing$ | 26 | 3808 |
| 61 | 0 | $\varnothing$ | $\{2\}$ | $\varnothing$ | $\varnothing$ | $\{0\}$ | 2 | 3722 |
| 62 | 5 | $\{1\}$ | $\varnothing$ | $\{-2,-1,0,1\}$ | $\varnothing$ | $\varnothing$ | 63 | 1787 |
| 63 | 4 | $\{1,3\}$ | $\varnothing$ | $\{-1,0,1,2\}$ | $\{3\}$ | $\varnothing$ | 40 | 1787 |
| 64 | 5 | $\{2\}$ | $\varnothing$ | $\{-2,-1,0,1\}$ | $\{0\}$ | $\varnothing$ | 31 | 1326 |
| 65 | 1 | $\varnothing$ | $\varnothing$ | $\{-2\}$ | $\{0\}$ | $\varnothing$ | 66 | 1055 |
| 66 | 2 | $\varnothing$ | $\varnothing$ | $\{-3,-2\}$ | $\varnothing$ | $\varnothing$ | 2 | 1055 |
| 67 | 4 | $\{2\}$ | $\varnothing$ | $\{0,1,2\}$ | $\{0,3\}$ | $\varnothing$ | 68 | 267 |
| 68 | 5 | $\{1,3\}$ | $\varnothing$ | $\{-1,0,1,2\}$ | $\{1\}$ | $\varnothing$ | 40 | 267 |
| 69 | 1 | $\varnothing$ | $\varnothing$ | $\{-3\}$ | $\varnothing$ | $\varnothing$ | 9 | 174 |

Table A2: Tuple digraph for $(1,-1)$GDN, part 2. Part 1 is Table A1.

| Prefix 1 | Prefix 2 | Suffix 1 | Suffix 2 | Count |
|---|---|---|---|---|
| 0101 | 0100100001 | 1010001010 | 1001000000100100 | 25742 |
| 0101 | 0100100000 | 1010001010 | 1001000000100100 | 64050 |
| 0101 | 0100001001 | 1010001010 | 1001000000100100 | 10741 |
| 0101 | 0100001000 | 1010001010 | 1001000000100100 | 71766 |
| 0101 | 0100000010 | 1010001010 | 1001000000100100 | 53427 |
| 010 | 01001000 | 10100010 | 1001000000100 | 159711 |
| 010 | 01000010 | 10100010 | 1001000000100 | 78945 |
| 010 | 01000000 | 10100010 | 1001000000100 | 8813 |
| 01 | 10010 | 10010 | 10000100 | 86230 |
| 01 | 10000 | 10010 | 10000100 | 7232 |
| 01 | 00100 | 10010 | 10000100 | 407 |
| 01 | 00010 | 10010 | 10000100 | 206340 |
| 01 | 00001 | 10010 | 10000100 | 59186 |
| 01 | 00000 | 10010 | 10000100 | 74524 |
| 00101010101 | 0100001000010010010000100000 | 1010010010100010100101001010 | 1001000010000100100000001001000010010000100100 | 4560 |
| 0010101010 | 0100001000010010010000000010 | 1010010010100010100101010010 | 10010000100001001000000010010000100100000100 | 9098 |
| 0010101001 | 0100100000001001000010001000 | 1010010010100010100100101010 | 10010000100001001000000010010000100000100100 | 454 |
| 0010101001 | 0100100000001001000010000010 | 1010010010100010100100101010 | 10010000100001001000000010010000100000100100 | 7995 |
| 0010101001 | 0100001000010010010000100001 | 1010010010100010100100101010 | 10010000100001001000000010010000100000100100 | 10391 |
| 0010101001 | 0100001000010010010000100000 | 1010010010100010100100101010 | 10010000100001001000000010010000100000100100 | 5566 |
| 001010100 | 0100100000001001000010010 | 1010010010100010100100010 | 1001000010000100100000001001000010000100 | 1258 |
| 001010100 | 0100001000010010010000010 | 1010010010100010100100010 | 1001000010000100100000001001000010000100 | 16314 |
| 0010101 | 010010010000100001 | 101001001000101010 | 1001000010000100000100100100 | 4832 |
| 0010101 | 010010010000100000 | 101001001000101010 | 1001000010000100000100100100 | 19122 |
| 0010101 | 010010010000001000 | 101001001000101010 | 1001000010000100000100100100 | 12215 |
| 0010101 | 010000100100100001 | 101001001000101010 | 1001000010000100000100100100 | 11714 |
| 0010101 | 010000100100100000 | 101001001000101010 | 1001000010000100000100100100 | 32402 |
| 0010101 | 010000100000010010 | 101001001000101010 | 1001000010000100000100100100 | 522 |
| 001010010101 | 010000100001001001001000001000 | 1010010010100101000010100101010 | 10010000100001001000010010000000010010000100100100 | 267 |
| 0010100101 | 01000010000100100001001001 | 10100100101001000100101010 | 100100001000010010000100000010000100100100 | 2239 |
| 001010010 | 010000100001001000010010 | 101001001010010001001010 | 1001000010000100100001000000100000100100 | 1483 |
| 001010 | 0100100100000010 | 1010010010001010 | 1001000010000100000100100 | 29508 |
| 001010 | 0100100001000000 | 1010010010001010 | 1001000010000100000100100 | 930 |
| 001010 | 0100001001001000 | 1010010010001010 | 1001000010000100000100100 | 16985 |
| 001010 | 0100001000000100 | 1010010010001010 | 1001000010000100000100100 | 154 |
| 001001010101 | 0100001001000010000100100100000 | 1010010010100100100010100101010 | 1001000010000100100001000010000001001000100100100 | 1787 |
| 00100101010 | 0100001001000010010010000010 | 10100100101001001000101001010 | 10010000100001001000010000100000010010000100100 | 7732 |
| 00100101001 | 0100001001001001000010000100 | 10100100101001000100101001010 | 10010000100001001000010000001000010010000100100 | 567 |
| 00100101001 | 0100001001001001000000100100 | 10100100101001000100101001010 | 10010000100001001000010000001000010010000100100 | 2139 |
| 00100101001 | 0100001001000010010010000100 | 10100100101001000100101001010 | 10010000100001001000010000001000010010000100100 | 1461 |
| 0010010100 | 0100001001001001000010000010 | 1010010010100100010010100 10 | 100100001000010010000100000010000100100000100 | 5908 |
| 0010010100 | 0100001001001001000000010010 | 10100100101001000100101010010 | 100100001000010010000100000010000100100000100 | 144 |
| 0010010100 | 0100001001000010010010000000 | 10100100101001000100101010010 | 100100001000010010000100000010000100100000100 | 3647 |
| 00100100101 | 0100001001001001000000100100 | 10100100101001000100100101010 | 10010000100001001000010000001000010000100100100 | 1326 |
| 001001001 | 01000010010000100100000010 | 1010010010100001010010 10 | 100100001000010010000000001001000010 0100 | 1234 |
| 0010010001010101 | 01000010010010010000000100100100100001000010 | 101001001010010010001010010010100101001010 | 100100001000010010000100001000000100100001000010010000100100000100100 | 1357 |
| 0010010001010101 | 01000010010010010000000100100100100001000000 | 101001001010010010001010010010100101001010 | 100100001000010010000100001000000100100001000010010000100100000100100 | 302 |
| 0010010001010101 | 01000010010010010000000100100100100000001000 | 101001001010010010001010010010100101001010 | 100100001000010010000100001000000100100001000010010000100100000100100 | 2149 |
| 00100100 | 0100001001000010010010 | 1010010010100001010010 | 1001000010000100100000000100100000100 | 358 |
| 001001 | 0100100100001001 | 1010010001001010 | 10010000100000010000100100 | 13793 |
| 001001 | 0100100100001000 | 1010010001001010 | 10010000100000010000100100 | 19404 |
| 001001 | 0100100100000010 | 1010010001001010 | 10010000100000010000100100 | 40187 |
| 001001 | 0100001000010010 | 1010010001001010 | 10010000100000010000100100 | 8008 |
| 0010001 | 0100001001001000010 | 1010010010001001010 | 10010000100001000000100001 00100 | 249 |
| 00100 | 01001001000010 | 10100100010010 | 10010000100000010000100 | 17609 |
| 00100 | 01001001000000 | 10100100010010 | 10010000100000010000100 | 3341 |
| 00010101 | 010000100100100100001 | 1010001010001010010 10 | 10010000001001000000100100001 00100 | 2629 |
| 00010101 | 010000100100100100000 | 101000101000101001010 | 1001000000100100000010010000100100 | 22759 |
| 00010101 | 010000100100001001001 | 101000101000101001010 | 1001000000100100000010010000100100 | 4109 |
| 00010101 | 010000100001001001001 | 101000101001000101010 | 1001000000100100001000000100100100 | 809 |
| 00010101 | 010000100001001000010 | 101000101001000101010 | 1001000000100100001000000100100100 | 246 |
| 0001010 | 0100001001001001000 | 1010001010001010010 | 1001000000100100000010010000100 | 4211 |
| 0001010 | 0100001001000010010 | 1010001010001010010 | 1001000000100100000010010000100 | 2829 |
| 000101 | 0100100100100001 | 1010001001001010 | 10010000001000010000100100 | 15232 |
| 000101 | 0100100100100000 | 1010001001001010 | 10010000001000010000100100 | 29000 |
| 000101 | 0100100001001001 | 1010001001001010 | 10010000001000010000100100 | 57205 |
| 000101 | 0100100001000010 | 1010001001001010 | 10010000001000010000100100 | 8840 |
| 0001001 | 0100001001001001001 | 1010001010001001010 | 1001000000100100000010000100100 | 174 |
| 00010 | 01001001001000 | 10100010010010 | 10010000001000010000100 | 35953 |
| 00010 | 01001000010010 | 10100010010010 | 10010000001000010000100 | 92 |
| 00010 | 01001000010000 | 10100010010010 | 10010000001000010000100 | 879 |
| 0 | 100 | 100 | 10000 | 567 |
| 0 | 000 | 100 | 10000 | 158374 |

Table A3: Multistring rewriting system for $(1, -1)$GDN.